DPNU-00-26
July 2000

On the DLCQ as a light-like limit in string theory

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Abstract

We study the issue of defining the discrete light-cone quantization (DLCQ) in perturbative string theory as a light-like limit. While this limit is unproblematic at the classical level, it is non-trivial at the quantum level due to the divergences by the zero-mode loops. We reconsider this problem in bosonic string theory. We construct the multi-loop scattering amplitudes in both open and closed string theories by using the method of Kikkawa, Sakita and Virasoro (KSV), and then we show that these scattering amplitudes are perfectly well-defined in this limit. We also discuss the vacuum amplitudes of the string theory. They are, however, ill-defined in the light-like limit due to the zero-mode loop divergences, and hence we want supersymmetry to cure those pathological divergences even in string theory.

1 Introduction

Matrix theory [1] is a supersymmetric quantum mechanics with matrix degrees of freedom. This was proposed as a hamiltonian of M-theory in the infinite momentum frame and it is expected to describe the fundamental degrees of freedom in the large N limit. Furthermore Susskind proposed that the discrete light-cone quantization (DLCQ) of M-theory is described by a finite N Matrix theory [2].

In DLCQ the compactified coordinate is the light-like coordinate $x^-(\simeq x^- + 2\pi R)$ and the corresponding momentum is quantized as $p^+ = N/R$. To prove the Susskind's conjecture, Seiberg described a light-like compactification as a limit of the compactification on a space-like circle [3, 4]. Following ref.[6], we will henceforth refer to the Seiberg's limit as a light-like limit (L^3) to distinguish it from the conventional DLCQ [5]. (We will also refer to the conventional DLCQ as the direct DLCQ.)

Several people examined the L^3 of scattering amplitudes in field and string theories compactified on an almost light-like circle [6, 10] or, Lorentz-equivalently, a vanishingly small space-like circle with fixed N [7, 8, 9, 10]. They found that the limit is, in general, complicated in field theory, but it is well-defined at one-loop level in Type II string theory. The problem in field theory is that when any external momenta in the compact direction[1]

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[1] Any external lines must have non-vanishing momenta in the compact direction [9].

do not flow through the loop, the zero-mode contribution in the sum over the discrete loop momentum, which corresponds to the loop diagram of only zero-mode propagators, is divergent as $\delta(0)$. Hence we can call this divergence the zero-mode loop divergence. In Type II string theory this type of divergence does not occur at the perturbative one-loop level. It was first pointed out that it would be due to the existence of the winding modes [7], and then it was pointed out that it is because the closed string theory has only three-point vertex [9]. Furthermore the existence of the light-like limit in string theory to all orders in perturbation was discussed in ref.[8]. But it is not obvious whether the result of the one-loop scattering amplitude can be applied to multi-loop scattering amplitudes without keeping the integrals, or sums, over loop momenta. To make this point clear is one of the purposes of this paper.

The vacuum energy in a theory which includes gravity is the cosmological constant and, in string theory, we must also investigate the vacuum amplitudes in the L^3. This is also one of the purposes of this paper.

The plan of this paper is as follows. In section 2 we briefly discuss the relevant kinematics. In section 3, we review the L^3 in field theory and in section 4, we construct the bosonic string multi-loop scattering amplitudes by the method of Kikkawa, Sakita and Virasoro (KSV) [11]. We show that these amplitudes are well-defined in the L^3. In section 5 we investigate the vacuum amplitudes in the L^3. Section 6 is devoted to our conclusion and discussions. In appendices A and B, we compare N-point tree and one-loop amplitudes by the KSV method with those of the operator formalism, respectively. We clarify how the amplitudes include self-crossing lines.

2 Kinematics

We consider a 26-dimensional space-time coordinate system (x^0, x^1, x^i), $(i = 2, \cdots, 25)$. The space-like coordinate x^1 takes values on a circle of radius $R_s = \epsilon R$, where we have introduced a parameter ϵ to consider $\epsilon \to 0$ limit with R fixed, so that the corresponding momentum p^1 is quantized, while the other coordinates x^0, x^i are non-compact:

$$\begin{pmatrix} x^1 \\ x^0 \end{pmatrix} \simeq \begin{pmatrix} x^1 \\ x^0 \end{pmatrix} + \begin{pmatrix} 2\pi R_s \\ 0 \end{pmatrix}, \qquad x^i \simeq x^i, \qquad p^1 = \frac{N}{R_s}. \tag{2.1}$$

By the Lorentz boost with a boost parameter $\beta = \frac{1-\epsilon^2/2}{1+\epsilon^2/2}$ we get a Lorentz equivalent coordinates $\tilde{x}^\mu$ [3, 7]

$$\begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^0 \end{pmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^0 \end{pmatrix}. \tag{2.2}$$

In the new coordinate system $\tilde{x}^\mu$, it is convenient to define the light-cone coordinates $x^\pm = (\tilde{x}^0 \pm \tilde{x}^1)/\sqrt{2}$. If ϵ is very small, the boost is very large and then the periodicity of eq.(2.1) becomes

$$\begin{pmatrix} x^- \\ x^+ \end{pmatrix} \simeq \begin{pmatrix} x^- \\ x^+ \end{pmatrix} + \begin{pmatrix} 2\pi R \\ -\epsilon^2 \pi R \end{pmatrix}, \qquad \tilde{x}^i \simeq \tilde{x}^i, \qquad p^+ = \frac{N}{R} + O(\epsilon^2). \tag{2.3}$$

(Here we should notice that the integer N in (2.1) agrees with that in (2.3).) Then eq.(2.3) clearly shows that in the $\epsilon \to 0$ limit with the fixed R, an almost light-like circle becomes

exactly a light-like one with radius R, while the Lorentz equivalent space-like circle (2.1) has shrunk to zero size. We get a discrete light-like coordinate system in this limit, which we call the light-like limit (L^3). Then we will be able to define DLCQ by quantizing theory on an almost light-like circle or a vanishingly small space-like circle and taking the L^3. We should study whether this limit is really well-defined or not at the quantum level. Hence, following ref.[7, 8, 9], we shall consider amplitudes in field and string theories on $M_{25} \times S^1$ (radius $R_s = \epsilon R$) and study their $\epsilon \to 0$ limit.

3 The L^3 in field theory

In this section, we investigate the L^3 in field theory and review the problem. In order to make a clear comparison between field theory and string theory, we take 26-dimensional ϕ^3 scalar (tachyon) field theory, where the mass square of a field is $-\frac{1}{\alpha'}$.

We consider four-point one-loop scattering processes, where each external state has an incoming momentum $P_i\,(i = 1, \cdots, 4)$. (see Figure 1.) By using a parameter representation of the propagators, $\Delta^{-1} = \int_0^1 dx\, x^{\Delta-1}$, the scattering amplitude of Figure 1(a) is given by[2],

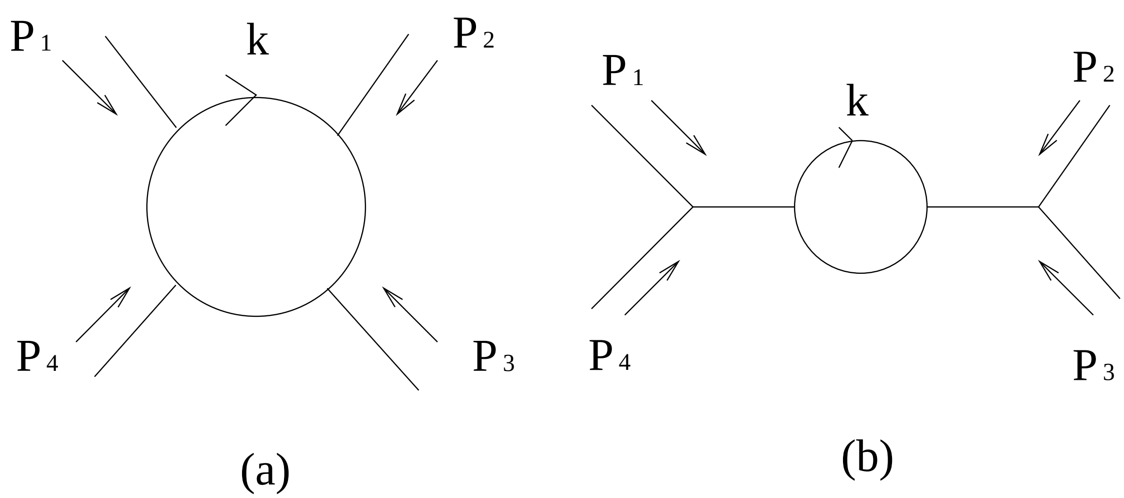

Figure 1: Two examples of four-point one-loop scattering processes in field theory.

$$
\begin{aligned}
A_{(a)} &= \frac{g^4}{\alpha'^4} \int d^{26}k \, \frac{1}{k^2 - \frac{1}{\alpha'}} \, \frac{1}{(k+P_2)^2 - \frac{1}{\alpha'}} \, \frac{1}{(k+P_2+P_3)^2 - \frac{1}{\alpha'}} \, \frac{1}{(k-P_1)^2 - \frac{1}{\alpha'}} \\
&= g^4 \int d^{26}k \int_0^1 \prod_{i=1}^{4} dx_i \, x_1^{\alpha' k^2 - 2} \, x_2^{\alpha'(k+P_2)^2-2} \, x_3^{\alpha'(k+P_2+P_3)^2-2} \, x_4^{\alpha'(k-P_1)^2-2}. \qquad (3.1)
\end{aligned}
$$

Here we introduce the external dual momenta $p_i\,(i = 1, \cdots, 4)$ as follows:

$$
\begin{aligned}
P_1 &= p_1 - p_4, \qquad (3.2) \\
P_2 &= p_2 - p_1, \qquad (3.3) \\
P_3 &= p_3 - p_2, \qquad (3.4) \\
P_4 &= p_4 - p_3. \qquad (3.5)
\end{aligned}
$$

[2]The signature of our metric is $\eta_{\mu\nu} = diag(-1, 1, 1, \cdots, 1)$.

Due to these definitions, the external momenta P_i automatically satisfy the conservation law $(\sum_{i=1}^{4} P_i = 0)$. Note that one of these p_i is, in fact, a redundant degree of freedom. By using these dual momenta and shifting the loop momentum $k \to -(k - p_1)$, we can rewrite (3.1) as follows:

$$A_{(a)} = g^4 \int d^{26}k \int_0^1 \prod_{i=1}^{4} dx_i \prod_{i=1}^{4} x_i^{-\alpha(-(k-p_i)^2)-1}, \tag{3.6}$$

where $\alpha(s)$ is the Regge trajectory function of open string,

$$\alpha(s) = \alpha' s + 1. \tag{3.7}$$

Similarly, the scattering amplitude corresponding to Figure 1(b),

$$A_{(b)} = \frac{g^4}{\alpha'^4} \int d^{26}k \; \frac{1}{(P_1+P_4)^2 - \frac{1}{\alpha'}} \frac{1}{k^2 - \frac{1}{\alpha'}} \frac{1}{(P_1+P_4-k)^2 - \frac{1}{\alpha'}} \frac{1}{(P_1+P_4)^2 - \frac{1}{\alpha'}}, \tag{3.8}$$

can be rewritten by

$$A_{(b)} = g^4 \int d^{26}k \int_0^1 dx_1 dx_3 dz_2 dz_4 \; x_1^{-\alpha(-(k-p_1)^2)-1} \, x_3^{-\alpha(-(k-p_3)^2)-1} \, (z_2 z_4)^{-\alpha(-(p_1-p_3)^2)-1}. \tag{3.9}$$

Now we consider the L^3 of these scattering amplitudes, i.e., we give the amplitudes on $M_{25} \times S^1$ (radius $R_s = \epsilon R$) and study their $\epsilon \to 0$ limit. On the $M_{25} \times S^1$ both the external and the loop momenta along the compact direction are quantized as $p_i^1 = n_i/R_s$, $k^1 = n/R_s$ and the integration over k^1 becomes a sum over n. Then the amplitude $A_{(a)}$ in (3.6) is given by,

$$\begin{aligned} A_{(a)} &= g^4 \int d^{25}k' \; \frac{1}{R_s} \sum_n \int_0^1 \prod_{i=1}^{4} dx_i \, (\prod_{i=1}^{4} x_i)^{-2} \\ &\quad \times \exp\left[\alpha' \sum_{i=1}^{4} \left\{ \left(\frac{n - n_i}{R_s} \right)^2 + (k' - p_i')^2 \right\} \ln x_i \right], \end{aligned} \tag{3.10}$$

where $(p_i') = (p_i^0, p_i^2, p_i^3, \cdots, p_i^{25})$ and $(k') = (k^0, k^2, k^3, \cdots, k^{25})$. Completing the squares for n and k', we can integrate over k' and we get,

$$\begin{aligned} A_{(a)} &= g^4 \int_0^1 \prod_{i=1}^{4} dx_i (\prod_{i=1}^{4} x_i)^{-2} \left(\frac{\pi}{-\alpha' \ln(\prod_{i=1}^{4} x_i)} \right)^{\frac{25}{2}} \\ &\quad \times \frac{1}{R_s} \sum_n \; \exp\left[\alpha' \ln(\prod_{i=1}^{4} x_i) \left(\frac{n}{R_s} - \frac{\sum_{i=1}^{4} n_i \ln x_i}{R_s \ln(\prod_{i=1}^{4} x_i)} \right)^2 \right. \\ &\quad \left. + \frac{\alpha'}{\ln(\prod_{i=1}^{4} x_i)} \{ \ln x_1 \ln x_2 \, (p_1 - p_2)^2 + \cdots + \ln x_3 \ln x_4 \, (p_3 - p_4)^2 \} \right]. \end{aligned} \tag{3.11}$$

We find that the R_s dependent parts combine to give a δ-function in the L^3 [7, 9],

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon R}\exp\left[\alpha'\ln(\prod_{i=1}^{4}x_i)\left(\frac{n}{\epsilon R}-\frac{\sum\limits_{i=1}^{4}n_i\ln x_i}{\epsilon R\,\ln(\prod\limits_{i=1}^{4}x_i)}\right)^2\right]$$
$$=\left(\frac{\pi}{-\alpha'\ln(\prod\limits_{i=1}^{4}x_i)}\right)^{\frac{1}{2}}\delta\left(n-\frac{\sum\limits_{i=1}^{4}n_i\ln x_i}{\ln(\prod\limits_{i=1}^{4}x_i)}\right). \tag{3.12}$$

Note that $\ln(\prod\limits_{i=1}^{4}x_i)\leq 0$ since $0\leq x_i\leq 1\,(i=1,\cdots,4)$. Plugging the δ-function into (3.11) we can integrate over one of the parameters x_i and we can get a well-defined result which agrees with that in the direct DLCQ [7]. But this story breaks down if the δ-function (3.12) is $\delta(0)$ identically and we encounter a pathology in the L^3 [6]. Due to eq.(3.12), it is obvious that this pathological $\delta(0)$ appears iff $n_1=n_2=n_3=n_4=n$. But this condition corresponds to the kinematical situation that all the external momenta in the S^1 direction $P_i^1=\frac{N_i}{R_s}\,(i=1,\cdots,4)$ are zero due to eqs.(3.2)$\sim$(3.5). It is obvious that this pathology does not arise in this scattering process of Figure 1(a).[3]

Next we consider the L^3 of the amplitude $A_{(b)}$ (3.9). Similarly to eq.(3.11), $A_{(b)}$ (3.9) on the $M_{25}\times S^1$ is given by,

$$\begin{aligned}A_{(b)} &= g^4\int_0^1 dx_1dx_3dz_2dz_4(x_1x_3)^{-2}(z_2z_4)^{-\alpha(-(p_1-p_3)^2)-1}\left(\frac{\pi}{-\alpha'\ln(x_1x_3)}\right)^{\frac{25}{2}}\\ &\times\frac{1}{R_s}\sum_n\ \exp\left[\alpha'\ln(x_1x_3)\left(\frac{n}{R_s}-\frac{n_1\ln x_1+n_3\ln x_3}{R_s\ln(x_1x_3)}\right)^2\right.\\ &\left.\qquad+\frac{\alpha'}{\ln(x_1x_3)}\ln x_1\ln x_3(p_1-p_3)^2\right].\end{aligned} \tag{3.13}$$

Then we get the following δ-function in the L^3,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon R}\exp\left[\alpha'\ln(x_1x_3)\left(\frac{n}{\epsilon R}-\frac{n_1\ln x_1+n_3\ln x_3}{\epsilon R\ln(x_1x_3)}\right)^2\right]$$
$$=\left(\frac{\pi}{-\alpha'\ln(x_1x_3)}\right)^{\frac{1}{2}}\delta\left(n-\frac{n_1\ln x_1+n_3\ln x_3}{\ln(x_1x_3)}\right). \tag{3.14}$$

Here we see that $n_1=n_3$ will give $\delta(0)$ in the amplitude. This corresponds to the kinematical situation that the external momenta $P_i^1(=\frac{N_i}{R_s})$ satisfy $\frac{N_1}{R_s}+\frac{N_4}{R_s}=0$ and $\frac{N_2}{R_s}+\frac{N_3}{R_s}=0$ due to eqs.(3.2)$\sim$(3.5) and this means that any external momenta P_i^1 do not flow through the loop. In this case when $n=n_1$ in the sum over the discrete loop momentum, the divergence appears. In the original loop momentum language, this divergence is obviously due to the zero-mode loop since we have shifted the loop momentum $k\to-(k-p_1)$. Hence this pathology is the same as before [6].

[3] See the footnote 1.

In field theory, we must add all possible Feynman diagrams as Figure 1(a) and 1(b) to calculate scattering amplitudes. By the above consideration, it is understood that in the L^3 only the zero-mode loop contributions among them are divergent as $\delta(0)$. It is well known that in the direct DLCQ the zero-modes are not dynamical degrees of freedom and they are written by the non-zero modes due to a nonlinear operator equation (zero-mode constraint [5]). Thus the zero-mode loop contributions do not appear in the direct DLCQ.

4 Multi-loop scattering amplitudes and their L^3 in string theory

In this section we investigate the L^3 of the multi-loop scattering amplitudes in bosonic string theory. In order to investigate the zero-mode loop problem, we adopt the method of Kikkawa, Sakita and Virasoro [11] to construct scattering amplitudes since the integration over loop momenta are explicitly kept in the amplitudes. In this method, s-t channel dualities and crossing symmetries are explicitly taken into account and it is easy to investigate whether we can take the L^3 of these amplitudes or not. First we briefly review their method. We clarify the rules for self-crossing lines in loop amplitudes and give tachyon scattering amplitudes in bosonic string theory. Then we will show that the L^3 of these amplitudes are well-defined.

4.1 KSV method in bosonic open string theory

In this subsection, we review the method of Kikkawa, Sakita and Virasoro in bosonic open string theory [11].[4] And then we will extend this method to bosonic closed string theory in the next subsection.

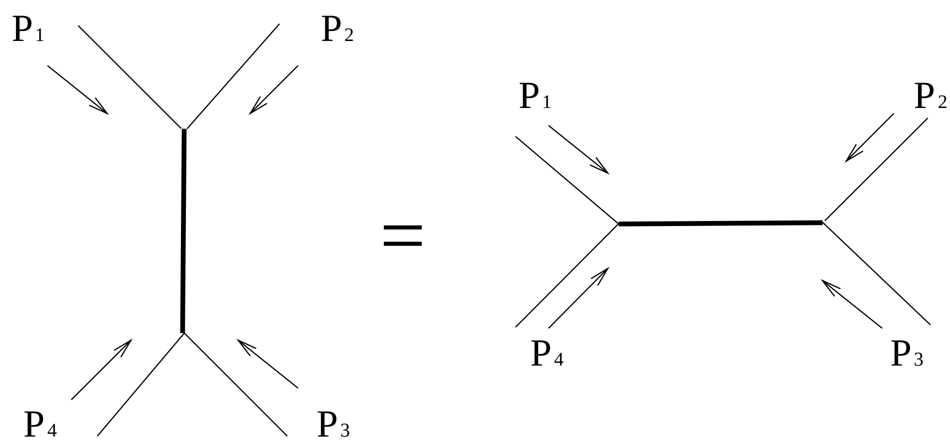

Figure 2: s-t channel duality in the Veneziano amplitude.

As is well known, the tachyon four-point tree amplitude (Veneziano amplitude) is given by,

$$A_4^{(0)} = g^2 \int_0^1 dx\, x^{-\alpha(s)-1} y(x)^{-\alpha(t)-1}, \tag{4.1}$$

[4]For simplicity, we restrict ourselves to the four-point planer amplitude in this subsection. We will extend this result N-point ones in Appendices A and B.

where

$$y(x) = 1 - x, \quad (4.2)$$

$$s = -(P_1 + P_2)^2, \quad (4.3)$$

$$t = -(P_2 + P_3)^2, \quad (4.4)$$

and $\alpha(s)$ is given in eq.(3.7) and P_i ($P_i^2 = \frac{1}{\alpha'}$, $i = 1, \cdots, 4$) are incoming external tachyon momenta. This amplitude manifestly has s-t channel duality (Figure 2). Here we consider the dual diagram to each diagram in Figure 2; i.e., we consider the quadrilateral defined by four sides which are dual to four external lines and two diagonals which are dual to the s- and t-channel propagators, respectively (Figure 3). Since the parameter of a propagator, e.g., y, corresponds to a line in the dual diagram, we simply call it the line y. We assign dual external momenta p_i ($i = 1, \cdots, 4$), which are defined by eqs.(3.2)$\sim$(3.5), to the four vertices of the quadrilateral in Figure 3. With these dual momenta, eqs.(4.3) and (4.4) are rewritten by,

$$s = -(p_2 - p_4)^2, \quad (4.5)$$

$$t = -(p_1 - p_3)^2. \quad (4.6)$$

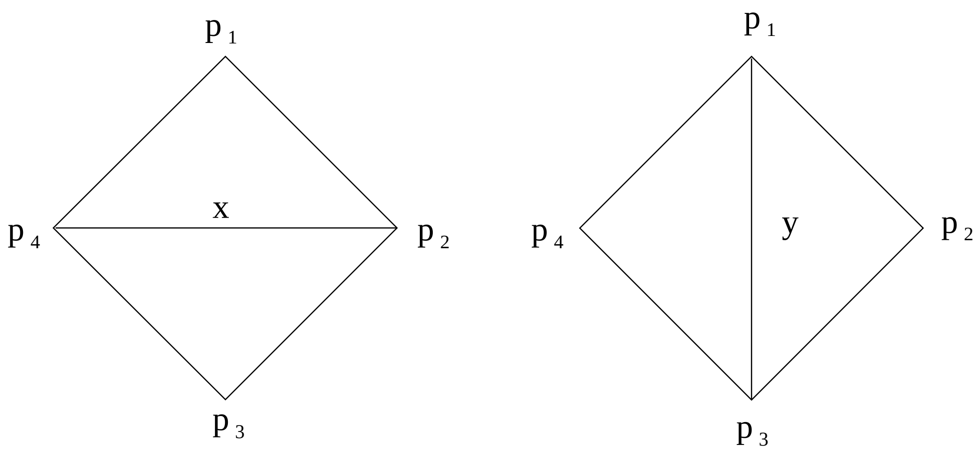

Figure 3: Two dual diagrams for the Veneziano amplitude.

Next we consider the four-point one-loop amplitude. The amplitude should contain the resonances in various field theoretical Feynman diagrams which are connected by s-t channel duality (Figure 4).

In order to construct the one-loop amplitude, we also consider the dual diagrams to those in Figure 4; i.e., we consider the quadrilateral defined by four sides which are dual to four external lines and an internal point which is dual to a closed loop and all possible lines which can be drawn between any two points of this quadrilateral, which are dual to all possible propagators (Figure 5). In each dual diagram, we always have four lines which connect two points of four vertices and one internal point and the four lines correspond to four propagators. Two lines which cross each other correspond to channels where we can not simultaneously find resonances. They have the same relation as the s-t channel in the Veneziano amplitude. Here, we can write the one-loop amplitude schematically as (4.1):

$$A_4^{(1)} \sim g^4 \int d^{26}k \int_0^1 \prod_{i=1}^{4} dx_i \prod_{j=1}^{k} y_j(x_1, \cdots, x_4)^{-\alpha(s_j)-1}, \quad (4.7)$$

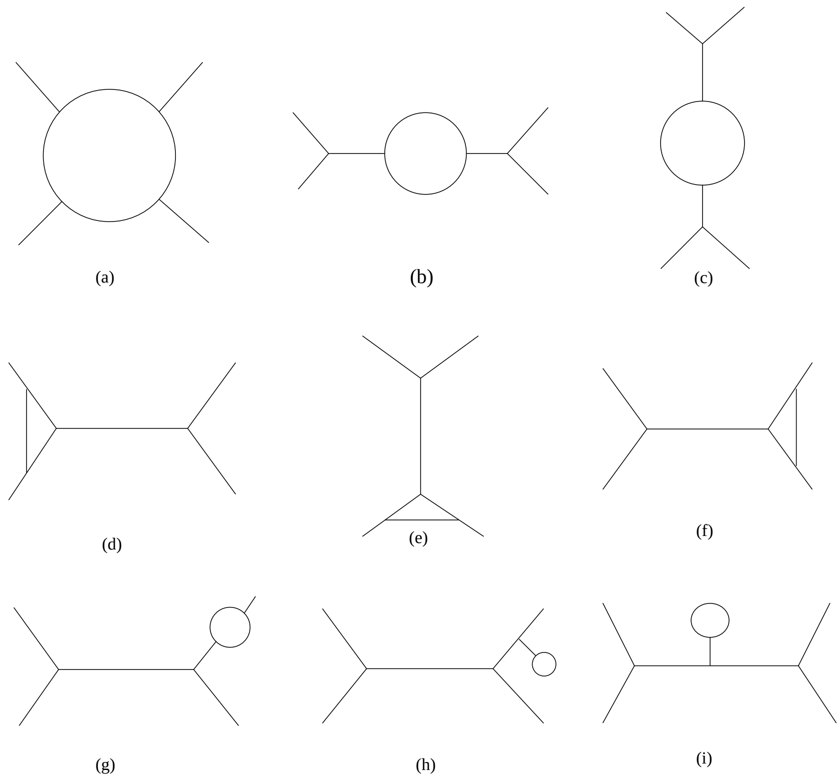

Figure 4: Examples of various one-loop field theoretical Feynman diagrams that are connected by s-t channel duality.

where factors $y_j(x_1, \cdots, x_4)$ are parameters of all possible propagators corresponding to all possible lines, which join pairs of points and do not cross themselves in dual diagrams and four of those are independent parameters $x_i\,(i = 1, \cdots, 4)$ and $-s_j$ are square of momenta carried by the j-th propagator. But this is not the end of the story. In order to obtain the correct one-loop amplitude, we must add the contributions of lines which cross themselves in dual diagrams, e.g., $z_3^{(1)}$ in Figure 6 (b), which do not correspond to any propagators in the field theoretical Feynman diagrams [11, 12]. Hence, the k factors $y_j(x_1, \cdots, x_4)$ should correspond to all possible topologically inequivalent lines that join pairs of points in dual diagrams. Since the k lines are not independent and we choose four independent lines $x_i\,(i = 1, \cdots, 4)$ in Figure 6 (a). Then all other lines $z_i^{(n)}$, $u_i^{(n)}\,(i = 1, \cdots, 4,\ n \geq 0)$ and $y_{12}^{(n)}$, $y_{23}^{(n)}$, $y_{34}^{(n)}$, $y_{41}^{(n)}\,(n \geq 0,\ n \leq -2)$ are functions of x_i as $y(x)$ in eq.(4.2). We draw some dependent lines in Figure 6 (b),(c),(d). We can determine these functions and we obtain the one-loop amplitude as follows:

$$\begin{aligned} A_4^{(1)} &= g^4 \int d^{26}k \int_0^1 \prod_{i=1}^{4} dx_i\, \rho(x_i) \prod_{i=1}^{4} x_i^{-\alpha(-(k-p_i)^2)-1} \left(\prod_{n=0}^{\infty} z_1^{(n)} z_3^{(n)} \right)^{-\alpha(-(p_2-p_4)^2)-1} \\ &\quad \times \left(\prod_{n=0}^{\infty} z_2^{(n)} z_4^{(n)} \right)^{-\alpha(-(p_1-p_3)^2)-1} \left(\prod_{n=0}^{\infty} u_1^{(n)} u_2^{(n)} u_3^{(n)} u_4^{(n)} \right)^{-\alpha(0)-1} \\ &\quad \times \left(\prod_{n=0}^{\infty} y_{12}^{(n)} y_{23}^{(n)} y_{34}^{(n)} y_{41}^{(n)} \prod_{n=2}^{\infty} y_{12}^{(-n)} y_{23}^{(-n)} y_{34}^{(-n)} y_{41}^{(-n)} \right)^{-\alpha(-\frac{1}{\alpha'})-1}, \end{aligned} \tag{4.8}$$

where $\rho(x_i)$ is the measure of the integral, which will be determined later. Note that the

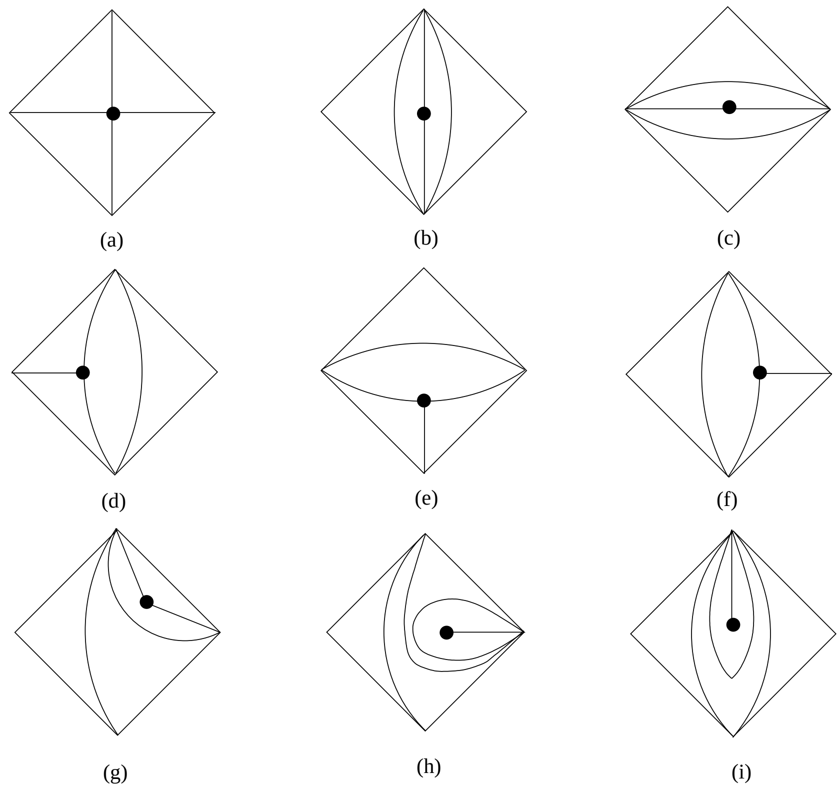

Figure 5: Examples of various dual diagrams for the one-loop field theoretical Feynman diagrams in Figure 4.

power of each line depends on the momentum of the corresponding propagator but it does not depend on n, i.e., how many times the line crosses itself. We should notice that if we put all the dependent parameters $z_i^{(n)}(x_i)$, $u_i^{(n)}(x_i)$, $y_{ij}^{(n)}(x_i)$ and measure $\rho(x_i)$ equal to one, this amplitude (4.8) is reduced to the one in the field theory (3.6).

Next we determine the forms of the functions, $z_i^{(n)}(x_i), u_i^{(n)}(x_i), y_{ij}^{(n)}(x_i)$ as $y(x)$ in eq.(4.2). For example, the lines x and y in eq.(4.2) are two diagonals that cross each other in the quadrilateral whose four sides correspond to external lines. Here we assume that even if four sides of the quadrilateral do not correspond to external lines, i.e., four sides have the parameters $a_i\,(i = 1, \cdots, 4)$ (Figure 7), one diagonal y is described by the function of the other diagonal x and four sides $a_i\,(i = 1, \cdots, 4)$ as follows:

$$y = f(x; a_1, a_2, a_3, a_4). \tag{4.9}$$

Of course, if all the sides of the quadrilateral correspond to the external lines; i.e., $a_1 = a_2 = a_3 = a_4 = 0$, eq.(4.9) should be reduced to (4.2). Under this assumption, for example, the line $z_1^{(0)}$ in Figure 6 (b), which is a diagonal of the quadrilateral whose sides are $(x_4,\, 0,\, 0,\, x_2)$, is described by[5],

$$z_1^{(0)} = f(x_1; x_4, 0, 0, x_2). \tag{4.10}$$

And $u_1^{(0)}$ and $y_{34}^{(0)}$, in Figure 6 (c), (d) are also given by,

$$u_1^{(0)} \;=\; f(x_3; x_1, z_2, z_4, x_1), \tag{4.11}$$

[5]A line may be regarded as the diagonal of different quadrilaterals. Uniqueness of the expression of a diagonal in this prescription is proved in the appendix B of [11].

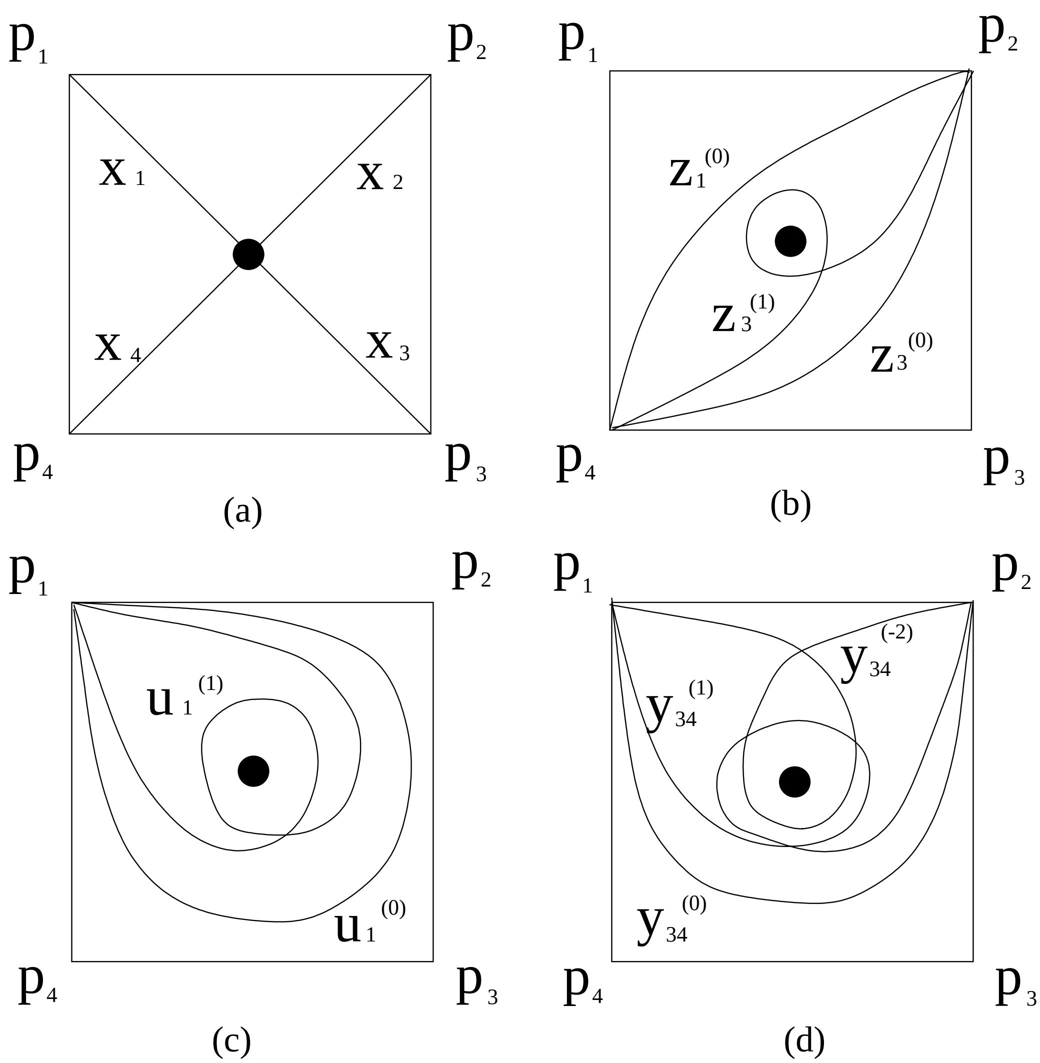

Figure 6: Some lines which must be added in one-loop dual diagrams; $y_{34}^{(0)}$ corresponds to a self-energy correction to the external line P_2, and $u_1^{(0)}$ is a tadpole line. Here, we should notice that the line $y_{34}^{(-1)}$ corresponds to the external line P_2 itself. Therefore, it is not included in the amplitude (4.8).

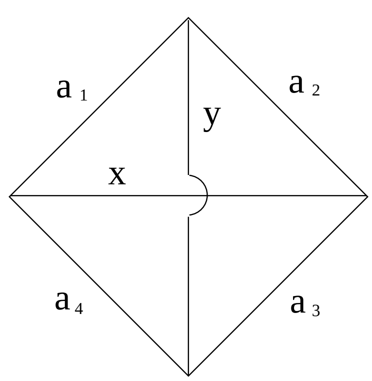

Figure 7: A part of the duality diagram. Line y is determined by a_1, a_2, a_3, a_4 and x.

$$y_{34}^{(0)} \;=\; f(x_4; x_2, z_3, 0, x_1). \tag{4.12}$$

Furthermore, we assume that n times winding (self-crossing) lines, $z_i^{(n)}$, $u_i^{(n)}$ and $y_{ij}^{(n)}$, are essentially the same diagonals as $z_i^{(0)}$, $u_i^{(0)}$ and $y_{ij}^{(0)}$, respectively, but since they cross all the $x_i (i = 1, \cdots, 4)$ n times, the corresponding diagonals, x_1, x_3 and x_4, should be multiplied by $(x_1x_2x_3x_4)^n$. Then we have

$$\begin{aligned} z_1^{(n)} &= f(x_1 w^n; x_4, 0, 0, x_2), && (4.13)\\ u_1^{(n)} &= f(x_3 w^n; x_1, z_2, z_4, x_1), && (4.14)\\ y_{34}^{(n)} &= f(x_4 w^n; x_2, z_3, 0, x_1), && (4.15)\\ w &\equiv x_1x_2x_3x_4. && (4.16) \end{aligned}$$

Now the remaining problem is to determine one function $y = f(x; a_1, a_2, a_3, a_4)$ [11]. In fact, we can determine this function by comparing the result of $N\,(\geq 8)$-point tree amplitude in the operator formalism with that by the KSV method. In appendix A, we show this in some details. The function $y = f(x; a_1, a_2, a_3, a_4)$ is given by [11],

$$y = f(x; a_1, a_2, a_3, a_4) = \frac{1 - x\alpha_2\alpha_3}{1 - x\alpha_2\alpha_3 a_1}\,\frac{1 - x\alpha_2\alpha_3 a_1 a_4}{1 - x\alpha_2\alpha_3 a_4}, \tag{4.17}$$

where

$$\begin{aligned} a_2 &= \frac{1-\alpha_2}{1-\alpha_2 a_1}\,\frac{1-\alpha_2 a_1 x}{1-\alpha_2 x}\,, && (4.18)\\ a_3 &= \frac{1-\alpha_3}{1-\alpha_3 a_4}\,\frac{1-\alpha_3 a_4 x}{1-\alpha_3 x}\,. && (4.19) \end{aligned}$$

Here we should notice that when all sides are external lines, $a_i = 0$, eq.(4.17) is reduced to $y = f(x; 0, 0, 0, 0) = 1 - x$.

Since the function has been determined, the one-loop scattering amplitude (4.8) is also determined up to the measure function $\rho(x_i)$. By the requirement of crossing symmetry in the amplitude (4.8), we can determine $\rho(x_i)$ up to an invariant function [11]:

$$\rho(x_i) \propto \frac{1}{(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_1)}. \tag{4.20}$$

In appendix B, by comparing the result of one-loop amplitudes in the operator formalism with those by the KSV method, we have determined the invariant function and then $\rho(x_i)$ is given by,

$$\begin{aligned} \rho(x_i) &= \frac{1}{(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_1)}\,[f(w)]^{-24}\,, && (4.21)\\ f(w) &\equiv \prod_{n=1}^{\infty}(1-w^n)\,. && (4.22) \end{aligned}$$

In conclusion, by this KSV method, we can construct the one-loop scattering amplitude, which is explicitly s-t channel dual and crossing symmetric by construction, up to $[f(w)]^{-24}$ factor. In appendix B, we show that the N-point one-loop amplitudes constructed by this KSV method agree with those in the operator formalism.

From the results of four-point tree and one-loop amplitudes, the extension to the four-point N-loop amplitude is obvious. In the N-loop case, we should consider the dual quadrilateral which has N internal points and choose $3N+1$ independent lines $x_j\,(j=1,\cdots,3N+1)$ among all the possible topologically inequivalent lines which connect any two points in the dual diagrams. Then the general form of four-point N-loop amplitude is given by,

$$
\begin{aligned}
A_4^{(N)} &= g^{2N+2}\int\prod_{J=1}^{N} d^{26}k_J \int_0^1 \prod_{j=1}^{3N+1} dx_j\, G\, D^{-\alpha(-(p_1-p_3)^2)-1}\, C^{-\alpha(-(p_2-p_4)^2)-1} \\
&\quad \times(\prod_{J=1}^{N} A_J^{-\alpha(-(k_J-p_2)^2)-1})\,(\prod_{J=1}^{N} A_J'^{-\alpha(-(k_J-p_4)^2)-1})\,(\prod_{J=1}^{N} B_J^{-\alpha(-(k_J-p_1)^2)-1}) \\
&\quad \times(\prod_{J=1}^{N} B_J'^{-\alpha(-(k_J-p_3)^2)-1})\,(\prod_{J<K} X_{JK}^{-\alpha(-(k_J-k_K)^2)-1})\,, \qquad (4.23)
\end{aligned}
$$

where A_J, A'_J, B_J, B'_J and $X_{JK}\,(J,K=1,\cdots,N)$ are the products of all parameters, dependent as well as independent, which correspond to topologically inequivalent lines connecting the internal point of the loop momentum k_J with those of momenta, p_2, p_4, p_1, p_3 and k_K, respectively. Furthermore C is the product of lines between p_2 and p_4 and D is the product of those between p_1 and p_3. G is a function of the parameters x_j and it is independent of both internal $k_J\,(J=1,\cdots,N)$ and external $p_i\,(i=1,\cdots,4)$ momenta.

4.2 KSV method in bosonic closed string theory

The KSV method is applicable to the bosonic closed string. In closed string amplitudes, we should change all the real parameters of propagators in open string amplitudes for complex ones and sum up all the terms corresponding to the different orderings of the external lines. For example, the closed string four-point tree amplitude (Shapiro-Virasoro amplitude), which corresponds to (4.1) in open string, is given by,

$$
\begin{aligned}
I_4^{(0)} &= \frac{\kappa^2}{4\pi}\int_{|x|\le 1} d^2x\,|x|^{-\alpha_C(s)-2}\,|1-x|^{-\alpha_C(t)-2} + (P_2\leftrightarrow P_3) \qquad (4.24)\\
&= \frac{\kappa^2}{4\pi}\int_{\mathbf{C}} d^2x\,|x|^{-\alpha_C(s)-2}\,|1-x|^{-\alpha_C(t)-2}, \qquad (4.25)
\end{aligned}
$$

where P_i ($P_i^2=\frac{4}{\alpha'}$, $i=1,\cdots,4$) are incoming external tachyon momenta and the Regge trajectory function α_C of closed string is given by,

$$
\alpha_C(s) = \frac{\alpha'}{2}s - 2. \qquad (4.26)
$$

In eq.(4.24) the first and second terms correspond to the different orderings of the external lines and by combining these two terms we can obtain the complete amplitude which has s-t-u channel duality [13]. For simplicity, however, we will henceforth consider only the term which correspond to the ordering $P_1\to P_2\to P_3\to P_4$ as the first term in eq.(4.24).

By the above considerations, closed string four-point one-loop amplitude, which corresponds to (4.8), is given by,

$$
I_4^{(1)} = \left(\frac{\kappa}{4\pi}\right)^4 \int d^{26}k \int_{|x_i|\le 1}\prod_{i=1}^{4} d^2x_i\,|\rho(x_i)|\,\prod_{i=1}^{4}|x_i|^{-\alpha_C(-(k-p_i)^2)-2}\left|\prod_{n=0}^{\infty} z_1^{(n)}\,z_3^{(n)}\right|^{-\alpha_C(-(p_2-p_4)^2)-2}
$$

$$\times \left|\prod_{n=0}^{\infty} z_2^{(n)} z_4^{(n)}\right|^{-\alpha_C(-(p_1-p_3)^2)-2} \left|\prod_{n=0}^{\infty} u_1^{(n)} u_2^{(n)} u_3^{(n)} u_4^{(n)}\right|^{-\alpha_C(0)-2}$$
$$\times \left|\prod_{n=0}^{\infty} y_{12}^{(n)} y_{23}^{(n)} y_{34}^{(n)} y_{41}^{(n)} \prod_{n=2}^{\infty} y_{12}^{(-n)} y_{23}^{(-n)} y_{34}^{(-n)} y_{41}^{(-n)}\right|^{-\alpha_C(-\frac{1}{\alpha'})-2}, \tag{4.27}$$

where
$$|\rho(x_i)| = \frac{1}{|(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_1)|^2}\,|f(w)|^{-48}, \tag{4.28}$$

which corresponds to (4.21). The complex parameters $z_i^{(n)}$, $u_i^{(n)}$ and $y_{ij}^{(n)}$ are given by the same functions of the complex parameters x_i as in (4.13)$\sim$(4.19) . In fact, it is easy to see that this one-loop amplitude agrees with that in the operator formalism in closed string theory. Furthermore, the closed string four-point N-loop amplitude, which corresponds to (4.23), is given by,

$$\begin{aligned} I_4^{(N)} &= \frac{\kappa^{2N+2}}{(4\pi)^{3N+1}} \int \prod_{J=1}^{N} d^{26}k_J \int_{|x_j|\le 1} \prod_{j=1}^{3N+1} d^2x_j\, |G||D|^{-\alpha_C(-(p_1-p_3)^2)-2}|C|^{-\alpha_C(-(p_2-p_4)^2)-2} \\ &\times (\prod_{J=1}^{N} |A_J|^{-\alpha_C(-(k_J-p_2)^2)-2})\,(\prod_{J=1}^{N} |A'_J|^{-\alpha_C(-(k_J-p_4)^2)-2})\,(\prod_{J=1}^{N} |B_J|^{-\alpha_C(-(k_J-p_1)^2)-2}) \\ &\times (\prod_{J=1}^{N} |B'_J|^{-\alpha_C(-(k_J-p_3)^2)-2})\,(\prod_{J<K} |X_{JK}|^{-\alpha_C(-(k_J-k_K)^2)-2})\,, \end{aligned} \tag{4.29}$$

where the complex functions $A_J, A'_J, B_J, B'_J, C, D, G$ and $X_{JK}\,(J,K = 1,\cdots,N)$ exactly correspond to those in open string, respectively.

4.3 The L^3 of multi-loop scattering amplitudes

In this subsection, we take the L^3 of amplitudes (4.23) and (4.29), i.e., we give the amplitudes on $M_{25}\times S^1$ (radius of S^1 is $R_s = \epsilon R$) and study their $\epsilon \to 0$ limit.

First, we consider the L^3 in open string theory. On the $M_{25}\times S^1$ both the external and the loop momenta along the compact direction are quantized as $p_i^1 = n_i/R_s$, $k_J^1 = n^{(J)}/R_s$ $(i = 1,\cdots,4,\ J = 1,\cdots,N)$ and the integration over k_J^1 becomes the sum over $n^{(J)}$, respectively. Then the amplitude $A_4^{(N)}$ in (4.23) is given by,

$$\begin{aligned} A_4^{(N)} &= g^{2N+2}\left(\frac{1}{R_s}\right)^N \sum_{n^{(1)},\cdots,n^{(N)}} \int \prod_{J=1}^{N} d^{25}k'_J \int_0^1 \prod_{j=1}^{3N+1} dx_j\, G\, D^{-\alpha(-(p_1-p_3)^2)-1}\, C^{-\alpha(-(p_2-p_4)^2)-1} \\ &\times \prod_{J=1}^{N}\Big(A_J A'_J B_J B'_J \prod_{K=J+1}^{N} X_{JK}\Big)^{-2} \exp\Bigg[\alpha' \sum_{J=1}^{N}\Bigg\{\Big(\big(k_J^1-p_2^1\big)^2 + (k'_J-p'_2)^2\Big)\ln A_J \\ &+ \Big(\big(k_J^1-p_4^1\big)^2 + (k'_J-p'_4)^2\Big)\ln A'_J + \Big(\big(k_J^1-p_1^1\big)^2 + (k'_J-p'_1)^2\Big)\ln B_J \\ &+ \Big(\big(k_J^1-p_3^1\big)^2 + (k'_J-p'_3)^2\Big)\ln B'_J + \sum_{K=J+1}^{N}\Big(\big(k_J^1-k_K^1\big)^2 + (k'_J-k'_K)^2\Big)\ln X_{JK}\Bigg\}\Bigg], \end{aligned} \tag{4.30}$$

where $(p'_i) = (p_i^0, p_i^2, p_i^3, \cdots, p_i^{25})$ and $(k'_J) = (k_J^0, k_J^2, k_J^3, \cdots, k_J^{25})$. Just as we completed the squares for n and k' in section 3, we successively complete the squares for N loop momenta,

$n^{(J)}$ and k'_J, starting with $n^{(N)}$ and k'_N, then $n^{(N-1)}$ and k_{N-1}, etc.. We can integrate over k'_J, however, in order to investigate whether we can take the L^3 of this amplitude or not, henceforth we only concentrate on the following R_s dependent part S_o of the amplitude,

$$
\begin{aligned}
S_o \;=\; & \left(\frac{1}{R_s}\right)^N \sum_{n^{(1)},\cdots,n^{(N)}} \exp\left[\alpha' \sum_{J=1}^{N} \left\{ \left(\frac{n^{(J)}}{R_s} - \frac{n_2}{R_s}\right)^2 \ln A_J + \left(\frac{n^{(J)}}{R_s} - \frac{n_4}{R_s}\right)^2 \ln A'_J \right.\right. \\
& + \left(\frac{n^{(J)}}{R_s} - \frac{n_1}{R_s}\right)^2 \ln B_J + \left(\frac{n^{(J)}}{R_s} - \frac{n_3}{R_s}\right)^2 \ln B'_J \\
& \left.\left. + \sum_{K=J+1}^{N} \left(\frac{n^{(J)}}{R_s} - \frac{n^{(K)}}{R_s}\right)^2 \ln X_{JK} \right\}\right]. \qquad (4.31)
\end{aligned}
$$

Completing the square for $n^{(N)}$, we obtain,

$$
\begin{aligned}
S_o \;=\; & \left(\frac{1}{R_s}\right)^N \sum_{n^{(1)},\cdots,n^{(N)}} \exp\left[\alpha' \ln\left(A_N A'_N B_N B'_N \prod_{K=1}^{N-1} X_{KN}\right)\right. \\
& \left.\times\left(\frac{n^{(N)}}{R_s} - \frac{n_2 \ln A_N + n_4 \ln A'_N + n_1 \ln B_N + n_3 \ln B'_N + \sum_{K=1}^{N-1} n^{(K)} \ln X_{KN}}{R_s \ln\left(A_N A'_N B_N B'_N \prod_{K=1}^{N-1} X_{KN}\right)}\right)^2\right] \\
& \times \exp\left[\alpha' \sum_{J=1}^{N-1} \left\{ \left(\frac{n^{(J)}}{R_s} - \frac{n_2}{R_s}\right)^2 \ln A_J^{(N-1)} + \left(\frac{n^{(J)}}{R_s} - \frac{n_4}{R_s}\right)^2 \ln A_J'^{(N-1)} \right.\right. \\
& + \left(\frac{n^{(J)}}{R_s} - \frac{n_1}{R_s}\right)^2 \ln B_J^{(N-1)} + \left(\frac{n^{(J)}}{R_s} - \frac{n_3}{R_s}\right)^2 \ln B_J'^{(N-1)} \\
& \left.\left. + \sum_{K=J+1}^{N-1} \left(\frac{n^{(J)}}{R_s} - \frac{n^{(K)}}{R_s}\right)^2 \ln X_{JK}^{(N-1)} \right\} + \cdots\right], \qquad (4.32)
\end{aligned}
$$

where $A_J^{(N-1)}, A_J'^{(N-1)}, B_J^{(N-1)}, B_J'^{(N-1)}$ $(1 \le J \le N-1)$ and $X_{IJ}^{(N-1)}$ $(1 \le I < J \le N-1)$ are given by,

$$
\ln A_J^{(N-1)} \;=\; \ln A_J + \frac{\ln A_N \ln X_{JN}}{\ln\left(A_N A'_N B_N B'_N \prod_{K=1}^{N-1} X_{KN}\right)}\,, \qquad (4.33)
$$

$$
\ln A_J'^{(N-1)} \;=\; \ln A'_J + \frac{\ln A'_N \ln X_{JN}}{\ln\left(A_N A'_N B_N B'_N \prod_{K=1}^{N-1} X_{KN}\right)}\,, \qquad (4.34)
$$

$$
\ln B_J^{(N-1)} \;=\; \ln B_J + \frac{\ln B_N \ln X_{JN}}{\ln\left(A_N A'_N B_N B'_N \prod_{K=1}^{N-1} X_{KN}\right)}\,, \qquad (4.35)
$$

$$
\ln B_J'^{(N-1)} \;=\; \ln B'_J + \frac{\ln B'_N \ln X_{JN}}{\ln\left(A_N A'_N B_N B'_N \prod_{K=1}^{N-1} X_{KN}\right)}\,, \qquad (4.36)
$$

$$\ln X_{IJ}^{(N-1)} = \ln X_{IJ} + \frac{\ln X_{IN} \ln X_{JN}}{\ln\left(A_N A'_N B_N B'_N \prod_{K=1}^{N-1} X_{KN}\right)} , \tag{4.37}$$

and "$\cdots$" denotes the terms which are independent of the loop momenta $n^{(J)}$. They play no essential role in taking the limit and henceforth we omit those terms. Note that the second exponential factor in eq.(4.32) takes the same form as that in eq.(4.31) up to a replacement $N \to N-1$. Furthermore, completing the squares for $n^{(N-1)}, n^{(N-2)}$, etc., successively, we obtain,

$$\begin{aligned}
S_o &= \left(\frac{1}{R_s}\right)^N \sum_{n^{(1)},\cdots,n^{(N)}} \exp\Bigg[\alpha' \ln\left(A_N A'_N B_N B'_N \prod_{K=1}^{N-1} X_{KN}\right) \\
&\times \left(\frac{n^{(N)}}{R_s} - \frac{n_2 \ln A_N + n_4 \ln A'_N + n_1 \ln B_N + n_3 \ln B'_N + \sum_{K=1}^{N-1} n^{(K)} \ln X_{KN}}{R_s \ln\left(A_N A'_N B_N B'_N \prod_{K=1}^{N-1} X_{KN}\right)}\right)^2\Bigg] \\
&\times \exp\Bigg[\alpha' \ln\left(A_{N-1}^{(N-1)} A'^{(N-1)}_{N-1} B_{N-1}^{(N-1)} B'^{(N-1)}_{N-1} \prod_{K=1}^{N-2} X_{KN-1}^{(N-1)}\right) \\
&\times \left(\frac{n^{(N-1)}}{R_s} - \frac{n_2 \ln A_{N-1}^{(N-1)} + \cdots + n_3 \ln B'^{(N-1)}_{N-1} + \sum_{K=1}^{N-2} n^{(K)} \ln X_{KN-1}^{(N-1)}}{R_s \ln\left(A_{N-1}^{(N-1)} A'^{(N-1)}_{N-1} B_{N-1}^{(N-1)} B'^{(N-1)}_{N-1} \prod_{K=1}^{N-2} X_{KN}^{(N-1)}\right)}\right)^2\Bigg] \\
&\quad \vdots \\
&\times \exp\Bigg[\alpha' \ln\left(A_1^{(1)} A'^{(1)}_1 B_1^{(1)} B'^{(1)}_1\right) \\
&\times \left(\frac{n^{(1)}}{R_s} - \frac{n_2 \ln A_1^{(1)} + n_4 \ln A'^{(1)}_1 + n_1 \ln B_1^{(1)} + n_3 \ln B'^{(1)}_1}{R_s \ln\left(A_1^{(1)} A'^{(1)}_1 B_1^{(1)} B'^{(1)}_1\right)}\right)^2\Bigg] .
\end{aligned} \tag{4.38}$$

Here, similarly to eqs.(4.33)$\sim$(4.37), $A_J^{(L)}, A'^{(L)}_J, B_J^{(L)}, B'^{(L)}_J$ $(1 \le J \le L \le N-1)$ and $X_{IJ}^{(L)}$ $(1 \le I < J \le L \le N-1)$ are given by,

$$\ln A_J^{(L)} = \ln A_J^{(L+1)} + \frac{\ln A_{L+1}^{(L+1)} \ln X_{JL+1}^{(L+1)}}{\ln\left(A_{L+1}^{(L+1)} A'^{(L+1)}_{L+1} B_{L+1}^{(L+1)} B'^{(L+1)}_{L+1} \prod_{K=1}^{L} X_{KL+1}^{(L+1)}\right)} , \tag{4.39}$$

$$\ln A'^{(L)}_J = \ln A'^{(L+1)}_J + \frac{\ln A'^{(L+1)}_{L+1} \ln X_{JL+1}^{(L+1)}}{\ln\left(A_{L+1}^{(L+1)} A'^{(L+1)}_{L+1} B_{L+1}^{(L+1)} B'^{(L+1)}_{L+1} \prod_{K=1}^{L} X_{KL+1}^{(L+1)}\right)} , \tag{4.40}$$

$$\ln B_J^{(L)} = \ln B_J^{(L+1)} + \frac{\ln B_{L+1}^{(L+1)} \ln X_{JL+1}^{(L+1)}}{\ln\left(A_{L+1}^{(L+1)} A'^{(L+1)}_{L+1} B_{L+1}^{(L+1)} B'^{(L+1)}_{L+1} \prod_{K=1}^{L} X_{KL+1}^{(L+1)}\right)} , \tag{4.41}$$

$$\ln B'^{(L)}_J = \ln B'^{(L+1)}_J + \frac{\ln B'^{(L+1)}_{L+1} \ln X_{JL+1}^{(L+1)}}{\ln\left(A_{L+1}^{(L+1)} A'^{(L+1)}_{L+1} B_{L+1}^{(L+1)} B'^{(L+1)}_{L+1} \prod_{K=1}^{L} X_{KL+1}^{(L+1)}\right)} , \tag{4.42}$$

$$\ln X_{IJ}^{(L)} = \ln X_{IJ}^{(L+1)} + \frac{\ln X_{IL+1}^{(L+1)} \ln X_{JL+1}^{(L+1)}}{\ln\left(A_{L+1}^{(L+1)} A_{L+1}^{\prime(L+1)} B_{L+1}^{(L+1)} B_{L+1}^{\prime(L+1)} \prod_{K=1}^{L} X_{KL+1}^{(L+1)}\right)}\ , \tag{4.43}$$

where $A_J^{(N)} = A_J$, $A_J^{\prime(N)} = A'_J$, $B_J^{(N)} = B_J$, $B_J^{\prime(N)} = B'_J$ and $X_{IJ}^{(N)} = X_{IJ}$. Similarly to eq.(3.12) in section 3, we find that S_o becomes to have N δ-functions in the L^3,

$$\begin{aligned}\lim_{\epsilon\to 0} S_o &= \sum_{n^{(1)},\cdots,n^{(N)}} \prod_{L=1}^{N} \left(\frac{\pi}{-\alpha' \ln\left(A_L^{(L)} A_L^{\prime(L)} B_L^{(L)} B_L^{\prime(L)} \prod_{K=1}^{L-1} X_{KL}^{(L)}\right)}\right)^{\frac{1}{2}} \\ &\times \prod_{L=1}^{N} \delta\left(n^{(L)} - \frac{n_2 \ln A_L^{(L)} + n_4 \ln A_L^{\prime(L)} + n_1 \ln B_L^{(L)} + n_3 \ln B_L^{\prime(L)} + \sum_{K=1}^{L-1} n^{(K)} \ln X_{KL}^{(L)}}{\ln\left(A_L^{(L)} A_L^{\prime(L)} B_L^{(L)} B_L^{(L)} \prod_{K=1}^{L-1} X_{KL}^{(L)}\right)}\right).\end{aligned} \tag{4.44}$$

Furthermore, after some calculations using eqs.(4.39)$\sim$(4.43), eq.(4.44) is rewritten in a more symmetrical form,

$$\begin{aligned}\lim_{\epsilon\to 0} S_o &= \sum_{n^{(1)},\cdots,n^{(N)}} \prod_{L=1}^{N} \left(\frac{-\pi \ln\left(A_L^{(L)} A_L^{\prime(L)} B_L^{(L)} B_L^{\prime(L)} \prod_{K=1}^{L-1} X_{KL}^{(L)}\right)}{\alpha'}\right)^{\frac{1}{2}} \\ &\times \prod_{L=1}^{N} \delta\Bigg(n^{(L)} \ln\left[A_L A'_L B_L B'_L \prod_{K=1}^{L-1} X_{KL} \prod_{K=L+1}^{N} X_{LK}\right] - n_2 \ln A_L - n_4 \ln A'_L \\ &\qquad -n_1 \ln B_L - n_3 \ln B'_L - \sum_{K=1}^{L-1} n^{(K)} \ln X_{KL} - \sum_{K=L+1}^{N} n^{(K)} \ln X_{LK}\Bigg).\end{aligned} \tag{4.45}$$

Then it is obvious that the condition $n_1 = n_2 = n_3 = n_4$, i.e., all the external momenta are zero, is necessary in order that the pathological $\delta(0)$ appears. Hence such a divergence never appears in open string multi-loop scattering amplitudes.

Next, we consider the L^3 in closed string theory. The story is essentially the same as open string theory, however, there are two new features in closed string theory, i.e., the existence of winding modes and the complexification of parameters.

On the $M_{25}\times S^1$, both the external and the loop momenta along the compact direction can have non-vanishing winding numbers. For simplicity, however, we assume that the external string states have vanishing winding numbers [7]. Under this assumption, the external and the loop momenta along the compact direction are written by,

$$p_i^1 = p_{Ri}^1 = p_{Li}^1 = \frac{n_i}{R_s}, \tag{4.46}$$

$$k_{RJ}^1 = \frac{n^{(J)}}{R_s} - \frac{m^{(J)} R_s}{\alpha'}, \tag{4.47}$$

$$k_{LJ}^1 = \frac{n^{(J)}}{R_s} + \frac{m^{(J)} R_s}{\alpha'}, \tag{4.48}$$

where the suffices R and L represent right and left movers of string states, respectively and $m^{(J)}$ is the winding number of the string in loop J. Furthermore, on the $M_{25} \times S^1$, the integration over k_J^1 becomes a sum over $n^{(J)}$ and we must also sum up over $m^{(J)}$. Then the amplitude $I_4^{(N)}$ in (4.29) is given by,

$$
\begin{aligned}
I_4^{(N)} &= \frac{\kappa^{2N+2}}{(4\pi)^{3N+1}} \left(\frac{1}{R_s}\right)^N \sum_{n^{(1)},\cdots,n^{(N)}} \sum_{m^{(1)},\cdots,m^{(N)}} \int \prod_{J=1}^{N} d^{25}k'_J \int_{|x_j|\le 1} \prod_{j=1}^{3N+1} d^2x_j \, |G| \\
&\quad \times |D|^{-\alpha_C(-(p_1-p_3)^2)-2} |C|^{-\alpha_C(-(p_2-p_4)^2)-2} \prod_{J=1}^{N} \left| A_J A'_J B_J B'_J \prod_{K=J+1}^{N} X_{JK} \right|^{-4} \\
&\quad \times \exp\left[\frac{\alpha'}{4} \sum_{J=1}^{N} \left\{ \left((k_{RJ}^1 - p_2^1)^2 + (k'_J - p'_2)^2 \right) \ln A_J + \left((k_{LJ}^1 - p_2^1)^2 + (k'_J - p'_2)^2 \right) \ln \bar{A}_J \right.\right. \\
&\qquad + \left((k_{RJ}^1 - p_4^1)^2 + (k'_J - p'_4)^2 \right) \ln A'_J + \left((k_{LJ}^1 - p_4^1)^2 + (k'_J - p'_4)^2 \right) \ln \bar{A}'_J \\
&\qquad + \left((k_{RJ}^1 - p_1^1)^2 + (k'_J - p'_1)^2 \right) \ln B_J + \left((k_{LJ}^1 - p_1^1)^2 + (k'_J - p'_1)^2 \right) \ln \bar{B}_J \\
&\qquad + \left((k_{RJ}^1 - p_3^1)^2 + (k'_J - p'_3)^2 \right) \ln B'_J + \left((k_{LJ}^1 - p_3^1)^2 + (k'_J - p'_3)^2 \right) \ln \bar{B}'_J \\
&\qquad + \sum_{K=J+1}^{N} \left((k_{RJ}^1 - k_{RK}^1)^2 + (k'_J - k'_K)^2 \right) \ln X_{JK} \\
&\qquad \left.\left. + \sum_{K=J+1}^{N} \left((k_{LJ}^1 - k_{LK}^1)^2 + (k'_J - k'_K)^2 \right) \ln \bar{X}_{JK} \right\} \right], \qquad (4.49)
\end{aligned}
$$

where $(p'_i) = (p_i^0, p_i^2, p_i^3, \cdots, p_i^{25})$ and $(k'_J) = (k_J^0, k_J^2, k_J^3, \cdots, k_J^{25})$. Similarly to eq.(4.31), henceforth we only concentrate on the following R_s dependent part S_c of the amplitude:

$$
\begin{aligned}
S_c &= \left(\frac{1}{R_s}\right)^N \sum_{n^{(1)},\cdots,n^{(N)}} \sum_{m^{(1)},\cdots,m^{(N)}} \exp\left[\frac{\alpha'}{4} \sum_{J=1}^{N} \left\{ \left(\frac{n^{(J)}}{R_s} - \frac{m^{(J)}R_s}{\alpha'} - \frac{n_2}{R_s} \right)^2 \ln A_J \right.\right. \\
&\quad + \left(\frac{n^{(J)}}{R_s} + \frac{m^{(J)}R_s}{\alpha'} - \frac{n_2}{R_s} \right)^2 \ln \bar{A}_J + \left(\frac{n^{(J)}}{R_s} - \frac{m^{(J)}R_s}{\alpha'} - \frac{n_4}{R_s} \right)^2 \ln A'_J \\
&\quad + \left(\frac{n^{(J)}}{R_s} + \frac{m^{(J)}R_s}{\alpha'} - \frac{n_4}{R_s} \right)^2 \ln \bar{A}'_J + \left(\frac{n^{(J)}}{R_s} - \frac{m^{(J)}R_s}{\alpha'} - \frac{n_1}{R_s} \right)^2 \ln B_J \\
&\quad + \left(\frac{n^{(J)}}{R_s} + \frac{m^{(J)}R_s}{\alpha'} - \frac{n_1}{R_s} \right)^2 \ln \bar{B}_J + \left(\frac{n^{(J)}}{R_s} - \frac{m^{(J)}R_s}{\alpha'} - \frac{n_3}{R_s} \right)^2 \ln B'_J \\
&\quad + \left(\frac{n^{(J)}}{R_s} + \frac{m^{(J)}R_s}{\alpha'} - \frac{n_3}{R_s} \right)^2 \ln \bar{B}'_J \\
&\quad + \sum_{K=J+1}^{N} \left(\left(\frac{n^{(J)}}{R_s} - \frac{m^{(J)}R_s}{\alpha'} - \frac{n^{(K)}}{R_s} + \frac{m^{(K)}R_s}{\alpha'} \right)^2 \ln X_{JK} \right. \\
&\qquad \left.\left.\left. + \left(\frac{n^{(J)}}{R_s} + \frac{m^{(J)}R_s}{\alpha'} - \frac{n^{(K)}}{R_s} - \frac{m^{(K)}R_s}{\alpha'} \right)^2 \ln \bar{X}_{JK} \right) \right\} \right] \\
&= \left(\frac{1}{R_s}\right)^N \sum_{n^{(1)},\cdots,n^{(N)}} \exp\left[\frac{\alpha'}{2} \sum_{J=1}^{N} \left\{ \left(\frac{n^{(J)}}{R_s} - \frac{n_2}{R_s} \right)^2 \mathrm{Re}\left(\ln A_J\right) \right.\right.
\end{aligned}
$$

$$+\left(\frac{n^{(J)}}{R_s}-\frac{n_4}{R_s}\right)^2 \mathrm{Re}\,(\ln A') + \left(\frac{n^{(J)}}{R_s}-\frac{n_1}{R_s}\right)^2 \mathrm{Re}\,(\ln B_J)$$
$$+\left(\frac{n^{(J)}}{R_s}-\frac{n_3}{R_s}\right)^2 \mathrm{Re}\,(\ln B'_J) + \sum_{K=J+1}^{N}\left(\frac{n^{(J)}}{R_s}-\frac{n^{(K)}}{R_s}\right)^2 \mathrm{Re}\,(\ln X_{JK})\Bigg\}\Bigg]\,\tilde{S}_c\,, \quad (4.50)$$

where,

$$\tilde{S}_c = \sum_{m^{(1)},\cdots,m^{(N)}} \exp\left[-\pi\,\vec{m}^T M^{-1}\vec{m} + 2\pi i\,\vec{m}^T\vec{x}\right], \quad (4.51)$$

$$M^{-1} = \frac{R_s^2}{2\pi\alpha'}\begin{bmatrix} a_{11} & a_{12} & & \cdots & a_{1N} \\ & a_{22} & a_{23} & \cdots & a_{2N} \\ & & \ddots & & \vdots \\ & 0 & & \ddots & a_{N-1N} \\ & & & & a_{NN}\end{bmatrix}, \quad (4.52)$$

$$\vec{m} = \begin{bmatrix} m^{(1)} \\ m^{(2)} \\ \vdots \\ m^{(N)}\end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(N)}\end{bmatrix}, \quad (4.53)$$

$$a_{JJ} = -\mathrm{Re}\left\{\ln\left(A_J A'_J B_J B'_J \prod_{K=1}^{J-1} X_{KJ} \prod_{K=J+1}^{N} X_{JK}\right)\right\}, \quad (4.54)$$

$$a_{JK} = 2\,\mathrm{Re}\,(\ln X_{JK}), \quad a_{KJ}=0, \qquad (J<K), \quad (4.55)$$

$$x^{(J)} = -\frac{1}{2\pi}\Bigg[n^{(J)}\mathrm{Im}\left\{\ln\left(A_J A'_J B_J B'_J \prod_{K=1}^{J-1} X_{KJ} \prod_{K=J+1}^{N} X_{JK}\right)\right\} - n_2\mathrm{Im}(\ln A_J)$$
$$-n_4\mathrm{Im}(\ln A'_J) - n_1\mathrm{Im}(\ln B_J) - n_3\mathrm{Im}(\ln B'_J)$$
$$-\sum_{K=1}^{J-1} n^{(K)}\mathrm{Im}(\ln X_{KJ}) - \sum_{K=J+1}^{N} n^{(K)}\mathrm{Im}(\ln X_{JK})\Bigg]. \quad (4.56)$$

Note that the (first) exponential factor in eq.(4.50) takes the same form as that in eq.(4.31) by replacing $\alpha' \to \frac{\alpha'}{2}$ and $\ln \to \mathrm{Re}\ln$. Hence, the remaining problem is to take the L^3 of $\tilde{S}_c$. This can be solved by using the Poisson resummation formula [7]:

$$\sum_{m^{(1)},\cdots,m^{(N)}} \exp\left[-\pi\vec{m}^T M^{-1}\vec{m} + 2\pi i\vec{m}^T\vec{x}\right] = \sum_{m^{(1)},\cdots,m^{(N)}} (\det M)^{\frac{1}{2}} \exp\left[-\pi(\vec{m}+\vec{x})^T M(\vec{m}+\vec{x})\right]. \quad (4.57)$$

Note that the $N\times N$ matrix M is proportional to $\frac{1}{R_s^2}$ due to eq.(4.52) and the $(\det M)^{\frac{1}{2}}$ on the r.h.s. gives $(\frac{1}{R_s})^N$ factor. Hence, after some calculations, we also obtain N δ-functions in the L^3:

$$\lim_{\epsilon\to 0}\tilde{S}_c = (2\pi)^N \sum_{m^{(1)},\cdots,m^{(N)}} \prod_{L=1}^{N} \delta\Bigg(n^{(L)}\mathrm{Im}\left\{\ln\left(A_L A'_L B_L B'_L \prod_{K=1}^{L-1} X_{KL} \prod_{K=L+1}^{N} X_{LK}\right)\right\}$$
$$-n_2\mathrm{Im}(\ln A_L) - n_4\mathrm{Im}(\ln A'_L) - n_1\mathrm{Im}(\ln B_L) - n_3\mathrm{Im}(\ln B'_L)$$
$$-\sum_{K=1}^{L-1} n^{(K)}\mathrm{Im}(\ln X_{KL}) - \sum_{K=L+1}^{N} n^{(K)}\mathrm{Im}(\ln X_{LK}) - 2\pi m^{(L)}\Bigg). \quad (4.58)$$

Thus, putting those together, we find that the S_c gives N complex δ-functions in the L^3,

$$\begin{aligned}
\lim_{\epsilon\to 0} S_c \;&=\; (2\pi)^N \sum_{n^{(1)},\cdots,n^{(N)}} \sum_{m^{(1)},\cdots,m^{(N)}} \prod_{L=1}^{N} \left(\frac{-2\pi \mathrm{Re} \ln \left(A_L^{(L)} A_L'^{(L)} B_L^{(L)} B_L'^{(L)} \prod_{K=1}^{L-1} X_{KL}^{(L)} \right)}{\alpha'} \right)^{\frac{1}{2}} \\
&\quad \times \prod_{L=1}^{N} \delta^{(2)} \left(n^{(L)} \ln \left(A_L A_L' B_L B_L' \prod_{K=1}^{L-1} X_{KL} \prod_{K=L+1}^{N} X_{LK} \right) - n_2 \ln A_L \right. \\
&\quad -n_4 \ln A_L' - n_1 \ln B_L - n_3 \ln B_L' \\
&\quad \left. - \sum_{K=1}^{L-1} n^{(K)} \ln X_{KL} - \sum_{K=L+1}^{N} n^{(K)} \ln X_{LK} - 2\pi i m^{(L)} \right). \qquad (4.59)
\end{aligned}$$

From the above equation, it is also obvious that $n_1 = n_2 = n_3 = n_4$ is a necessary condition for an appearance of the pathological $\delta(0)$. Hence such a divergence does not occur in closed string multi-loop scattering amplitudes.

We have examined the L^3 of multi-loop open and closed string scattering amplitudes. In conclusion, *since by construction these amplitudes consist of all the possible propagators in the field theoretical Feynman diagrams, which is the root of s-t channel duality, we can take the L^3 of these scattering amplitudes, i.e., the zero-mode loop divergences do not appear.* Note that it is obvious that the nonexistence of the $\delta(0)$ is not essentially due to the existence of the winding modes (cf. [7]).

5 The L^3 of vacuum amplitudes in string theory

In this section, we discuss the L^3 of vacuum amplitudes in bosonic string theory. For simplicity, we restrict ourselves to bosonic closed string theory.

In the previous section, we have seen that the factor $\frac{1}{R_s^2} \sim \frac{1}{\epsilon^2}$ has come out for each loop from the integration measure of the compactified loop momentum and the Poisson resummation of the winding modes. This factor has combined with an appropriate exponential function $\exp\left(-\left(\frac{\pi}{R_s^2}\right)|\cdots|^2\right)$ to give a complex δ-function for each loop. We have shown that this complex δ-function can become $\delta^{(2)}(0)$ iff all the external momenta in the compact direction $P_i^1 = \frac{N_i}{R_s}$ are zero at multi-loop order. Thus, this pathological $\delta^{(2)}(0) \sim \frac{1}{R_s^2}$ does not appear in the multi-loop string scattering amplitudes. As for the vacuum amplitudes which have no external lines, however, it is obvious that this pathological situation inevitably occurs, i.e., the zero-mode loop divergences always appear. In fact, as is well known, in a T-dual theory this $\delta^{(2)}(0) \sim \frac{1}{R_s^2}$ for each loop is absorbed in the T-dual coupling constant $\tilde{\kappa} = e^{\tilde{\phi}}$,

$$\tilde{\kappa}^2 = \kappa^2 \frac{\alpha'}{R_s^2}, \qquad (5.1)$$

where κ is the original closed string coupling constant. In the L^3, since we shall keep the original string coupling constant fixed, this dual coupling constant diverges. Thus the zero-modes become strongly coupled and the cosmological constant can not be determined perturbatively. In particular, since the closed string theory includes gravity, we shall take

this problem seriously. Of course, we should notice that the M-theory or superstring does not suffer from this problem due to supersymmetry, and neither does the direct DLCQ theory due to the triviality of the vacuum.

6 Conclusion and Discussion

In this paper, we have argued whether perturbative string multi-loop amplitudes have well-defined light-like limit or not. To answer this question, we have used the multi-loop string scattering amplitudes constructed by the method of [11]. In this method, since we can keep the integral, or sum, over the loop momenta, we have shown that the perturbative string multi-loop scattering amplitudes have well-defined light-like limit, which was conjectured in ref.[8]. This result depends on the striking stringy nature. In fact, we must add all possible Feynman diagrams in a certain order of perturbation theory and the zero-mode loop diagrams among them diverge as $\delta(0)$ in field theory, while in string theory, since a scattering amplitude in a certain order of perturbation contains the resonances in the all possible field theoretical Feynman diagrams in the same order of perturbation, the zero-mode loop divergences do not appear.

We also have discussed the light-like limit of the vacuum amplitudes in bosonic string theory. These amplitudes obviously diverge as $\delta(0)$ even in string theory. Therefore, we need supersymmetry for this limit to be well-defined. It seems that this result prevents us from extracting the information of supersymmetry breaking in M-theory or superstring from the finite N Matrix theory. In fact, it seems that the existence of non-supersymmetric Matrix string theory is questionable [14].

A KSV construction of N-point tree amplitude

In this appendix, we calculate the tachyon N-point tree amplitude in bosonic open string theory by the KSV method [11]. We shall determine the form of the function $y = f(x; a_1, a_2, a_3, a_4)$, i.e., (4.17)$\sim$(4.19), by comparing the eight-point amplitude by the KSV method with that in the operator formalism. Then by using this function, we will show that the N-point tree amplitude constructed by the KSV method agrees with the one in the operator formalism. Similarly to the four-point amplitude in section 4, we consider the dual diagrams to the field theoretical Feynman diagrams connected by s-t channel duality for the N-point tree amplitude. That is, we consider the N-polygons defined by N sides which are dual to the N external lines. Each diagonal of the N-polygon is dual to a propagator. In each dual diagram, we always have $N-3$ diagonals which correspond to the $N-3$ propagators in the field theoretical Feynman diagram.

We define the dual external momenta p_i $(i = 1, \cdots, N)$ as follows (see (3.2) $\sim$ (3.5)):

$$\begin{aligned} P_1 &= p_1 - p_N, \\ P_2 &= p_2 - p_1, \\ &\cdots \\ P_N &= p_N - p_{N-1}, \end{aligned} \tag{A.1}$$

and we assign p_i $(i = 1, \cdots, N)$ to the N vertices of the N-polygon, respectively (Figure 8). The $\frac{N(N-3)}{2}$ lines connecting the vertices p_{i-1} and p_j $(2 \le i < j \le N-1,\ 3 \le i < j = N)$

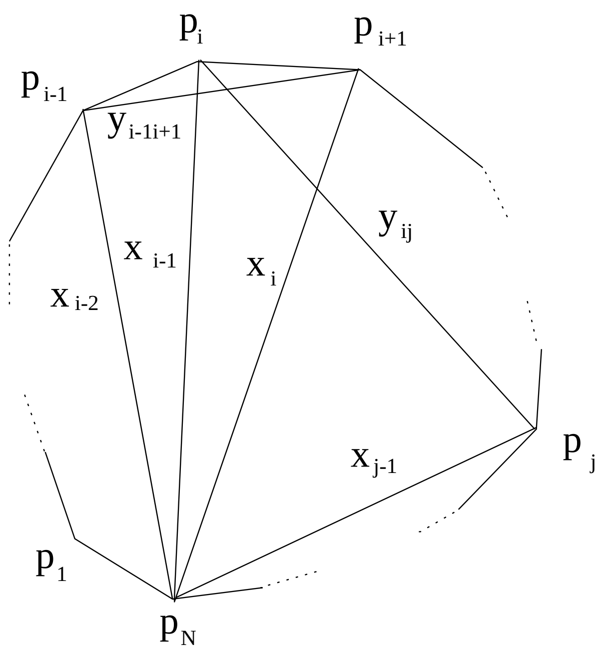

Figure 8: (Dual) diagram for the N-point tree amplitude.

are denoted by $y_{i-1\,j}$ and we choose $N-3$ independent lines $x_i \equiv y_{i+1N}$ $(i = 1, \cdots, N-3)$ among them (Figure 8). Then we can write down the N-point tree amplitude by the KSV method:

$$A_N^{(0)} = g^{N-2} \int_0^1 \prod_{i=1}^{N-3} dx_i\, \rho^{(0)}(x_i) \prod_{i=1}^{N-3} x_i^{-\alpha(-(p_{i+1}-p_N)^2)-1} \prod_{2\le i<j\le N-1} y_{i-1j}^{-\alpha(-(p_{i-1}-p_j)^2)-1}, \tag{A.2}$$

where $\rho^{(0)}(x_i)$ is determined by the requirement that the integral volume element maintains the crossing symmetry[15, 16],

$$\rho^{(0)}(x_i) = \frac{1}{\prod_{2\le i<N-2} (1 - x_i x_{i-1})}. \tag{A.3}$$

Next we shall determine the forms of the functions $y_{i-1j}(x_k)$ $(2 \le i < j \le N-1)$ and compare (A.2) with the tree amplitude in the operator formalism. In the operator formalism the N-point tree amplitude is given by [17],

$$A_{Nop}^{(0)} = g^{N-2} \int_0^1 \prod_{i=1}^{N-3} dx_i \prod_{i=1}^{N-3} x_i^{-\alpha(s_i)-1} \prod_{2\le i<j\le N-1} (1 - x_{ij})^{2\alpha'(P_i\cdot P_j)}, \tag{A.4}$$

where

$$\begin{aligned} x_{ij} &= x_{i-1}x_i \cdots x_{j-2}, && \text{(A.5)} \\ s_i &= -(P_1 + \ldots + P_{i+1})^2, && \text{(A.6)} \end{aligned}$$

From eqs.(A.1) we obtain,

$$\begin{aligned} (p_{i-1} - p_j)^2 &= (P_i + \ldots + P_j)^2, && \text{(A.7)} \\ s_i &= -(p_{i+1} - p_N)^2. && \text{(A.8)} \end{aligned}$$

Then, after some calculations, (A.4) is rewritten by,

$$
\begin{aligned}
A^{(0)}_{Nop} &= g^{N-2} \int_0^1 \prod_{i=1}^{N-3} dx_i \rho^{(0)}(x_i) \prod_{i=1}^{N-3} x_i^{-\alpha(-(p_{i+1}-p_N)^2)-1} \\
&\times \prod_{2\le i<j\le N-1} \left(\frac{1-x_{ij}}{1-x_{i-1j}} \frac{1-x_{i-1j+1}}{1-x_{ij+1}} \right)^{-\alpha(-(p_{i-1}-p_j)^2)-1} .
\end{aligned}
\tag{A.9}
$$

Here, we have used the equations which $x_0 = x_{N-2} = 0$, i.e., $x_{1j} = x_{iN} = 0$.

Now we concentrate on the $N = 8$ case and determine the form of the function (4.9), $y = f(x; a_1, a_2, a_3, a_4)$. We can easily see that (A.2) agrees with (A.9) if

$$
y_{i-1j} = \frac{1-x_{ij}}{1-x_{i-1j}} \frac{1-x_{i-1j+1}}{1-x_{ij+1}}. \tag{A.10}
$$

For example, the lines y_{26}, y_{24} and y_{46} are written by,

$$
y_{26} = \frac{1-x_2x_3x_4}{1-x_1x_2x_3x_4} \frac{1-x_1x_2x_3x_4x_5}{1-x_2x_3x_4x_5}, \tag{A.11}
$$

$$
y_{24} = \frac{1-x_2}{1-x_1x_2} \frac{1-x_1x_2x_3}{1-x_2x_3}, \tag{A.12}
$$

$$
y_{46} = \frac{1-x_4}{1-x_3x_4} \frac{1-x_3x_4x_5}{1-x_4x_5}. \tag{A.13}
$$

In Figure 9 we see that the line y_{26} is a diagonal of the quadrilateral whose sides are $(x_1, y_{24}, y_{46}, x_5)$ and the other diagonal of this quadrilateral is x_3. And hence we can write,

$$
y_{26} = f(x_3; x_1, y_{24}, y_{46}, x_5). \tag{A.14}
$$

Then we have obtained eqs.(4.17)$\sim$(4.19). Note that when $N < 8$, we cannot draw a quadrilateral whose all sides are diagonals of the N-polygon of the dual diagram.

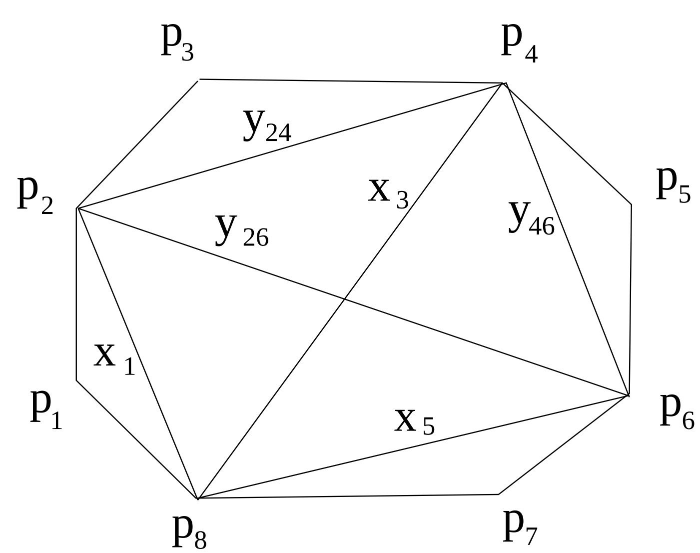

Figure 9: (Dual) diagram for the eight-point tree amplitude.

Next by using these eqs.(4.17)$\sim$(4.19), we show that (A.10) holds for an arbitrary N, i.e., the N-point tree amplitude by the KSV method agrees with that in the operator formalism. Let us consider the lines y_{i-1i+1} $(2 \leq i \leq N-2)$ in Figure 8. Since y_{i-1i+1} is a diagonal of the quadrilateral whose sides are $(x_{i-2}, 0, 0, x_i)$ with the other diagonal x_{i-1}, we have

$$\begin{aligned} y_{i-1i+1} &= f(x_{i-1}; x_{i-2}, 0, 0, x_i) \\ &= \frac{1-x_{i-1}}{1-x_{i-1}x_{i-2}} \frac{1-x_{i-2}x_{i-1}x_i}{1-x_{i-1}x_i}. \end{aligned} \tag{A.15}$$

Then we will prove (A.10) by mathematical induction [11]. Suppose the equation holds for y_{i-1j}. Since the line y_{i-1j+1} is a diagonal of the quadrilateral whose sides are $(x_{i-2}, y_{i-1j}, 0, x_j)$, we have

$$\begin{aligned} y_{i-1j+1} &= f(x_{j-1}; x_{i-2}, y_{i-1j}, 0, x_j) \\ &= \frac{1-x_{j-1}\alpha}{1-x_{j-1}\alpha x_{i-2}} \frac{1-x_{j-1}\alpha x_{i-2}x_j}{1-x_{j-1}\alpha x_j}. \end{aligned} \tag{A.16}$$

where α is defined implicitly by,

$$y_{i-1j} = \frac{1-\alpha}{1-\alpha x_{i-2}} \frac{1-\alpha x_{j-1}x_{i-2}}{1-\alpha x_{j-1}}\,. \tag{A.17}$$

Here by the assumption of mathematical induction, we obtain, $\alpha = x_{ij} = x_{i-1}\cdots x_{j-2}$. Plugging this into (A.16), we obtain $y_{i-1j+1} = \frac{1-x_{ij+1}}{1-x_{i-1j+1}} \frac{1-x_{i-1j+2}}{1-x_{ij+2}}$. Hence we have proven (A.10) for an arbitrary N.

B KSV construction of N-point one-loop amplitude

In this appendix we construct the tachyon N-point one-loop amplitude in bosonic open string theory by the KSV method [11]. Each dual diagram to the field theoretical Feynman diagram is a N-polygon with an internal point and N internal lines which connect either two of the vertices or a vertex and an internal point properly and do not intersect each other. The N sides of the polygon and the internal point are dual to the N external lines and the loop in the Feynman diagram, respectively. In order to construct the amplitude we shall consider all topologically inequivalent lines connecting either two of the vertices or a vertex and the internal point of the N-polygon. As in the previous appendix, we assign the dual momenta $p_i\,(i = 1, \cdots, N)\,(A.1)$ to the N vertices of the N-polygon and we call the internal point o_1. The N lines connecting the vertices p_i and the internal point o_1, which do not exist in the tree case, are denoted by x_i. As for the lines connecting two vertices of the N-polygon, a pair of vertices do not uniquely determine a line in this case due to the existence of the point o_1 and the key to constructing the one-loop amplitude is to completely determine those lines.

From p_r to p_s we draw a line clockwise with the point o_1 kept to the right. So we can draw two lines for each pair of the vertices p_r and p_s. One is from p_r to p_s and the other is from p_s to p_r. They are denoted by $y^{(0)}_{rs}$ and $y^{(0)}_{sr}$, respectively (see Figure 10). Note that without the point o_1, i.e., for the tree amplitude, $y^{(0)}_{rs}$ and $y^{(0)}_{sr}$ are (topologically) equivalent, but this is not the case with o_1. We should notice that $y^{(0)}_{rr+1}$ is a side of the polygon and

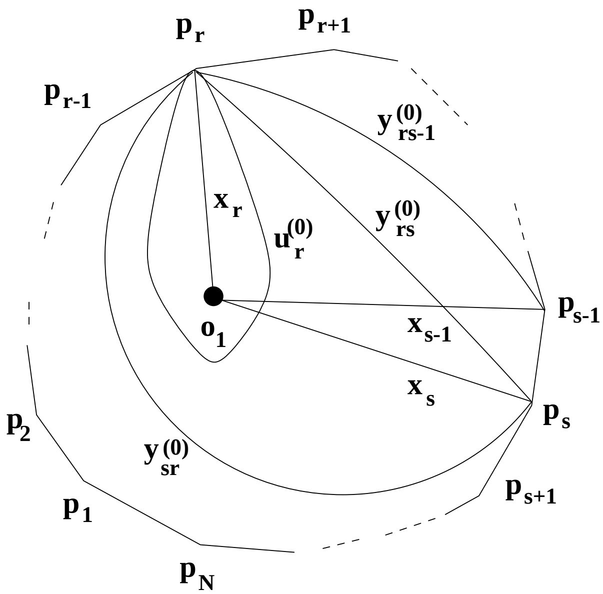

Figure 10: (Dual) diagram for the N-point one-loop amplitude.

hence $y^{(0)}_{rr+1} = 0$, but $y^{(0)}_{r+1r}$ is not. Furthermore we can draw a non-trivial line from p_r going around the point o_1 and coming back to p_r, which is denoted by $u^{(0)}_r$ (Figure 10).

As in the tree case, x_i, $y^{(0)}_{rs}, y^{(0)}_{sr}, y^{(0)}_{r+1r}$ and $u^{(0)}_r$ are not independent and we can take x_i ($i = 1, \cdots, N$) as independent parameters and determine $y^{(0)}_{rs}, y^{(0)}_{sr}, y^{(0)}_{r+1r}$ and $u^{(0)}_r$ as functions of x_i. We explain how to obtain $y^{(0)}_{rs} = y^{(0)}_{rs}(x_i)$ ($1 \le r < s-1 < N$, $(r,s) \neq (1,N)$) below. We shall see that $y^{(0)}_{rs}$ is a diagonal of a quadrilateral $o_1 p_r p_{s-1} p_s$ whose sides are $(x_r, y^{(0)}_{rs-1}, 0, x_s)$ and hence we have,

$$y^{(0)}_{rs} = f(x_{s-1}; x_r, y^{(0)}_{rs-1}, 0, x_s)\,. \tag{B.1}$$

where f is defined by (4.17)$\sim$(4.19). Similarly $y^{(0)}_{rs-1}$ is a diagonal of a quadrilateral $o_1 p_r p_{s-2} p_{s-1}$, $y^{(0)}_{rs-2}$ is a diagonal of a quadrilateral $o_1 p_r p_{s-3} p_{s-2}$ and so on. Then we have the following series of equations:

$$\begin{aligned}
y^{(0)}_{rs-1} &= f(x_{s-2}; x_r, y^{(0)}_{rs-2}, 0, x_{s-1})\,, && \text{(B.2)}\\
y^{(0)}_{rs-2} &= f(x_{s-3}; x_r, y^{(0)}_{rs-3}, 0, x_{s-2})\,, && \text{(B.3)}\\
&\cdots\\
y^{(0)}_{rr+4} &= f(x_{r+3}; x_r, y^{(0)}_{rr+3}, 0, x_{r+4})\,, && \text{(B.4)}\\
y^{(0)}_{rr+3} &= f(x_{r+2}; x_r, y^{(0)}_{rr+2}, 0, x_{r+3})\,, && \text{(B.5)}\\
y^{(0)}_{rr+2} &= f(x_{r+1}; x_r, 0, 0, x_{r+2})\,. && \text{(B.6)}
\end{aligned}$$

Eq.(B.5) gives,

$$\begin{aligned}
y^{(0)}_{rr+3} &= \frac{1 - x_{r+2}\alpha}{1 - x_{r+2}\alpha x_r}\,\frac{1 - x_{r+2}\alpha x_r x_{r+3}}{1 - x_{r+2}\alpha x_{r+3}}, && \text{(B.7)}\\
y^{(0)}_{rr+2} &= \frac{1-\alpha}{1-\alpha x_r}\,\frac{1-\alpha x_r x_{r+2}}{1-\alpha x_{r+2}}\,, && \text{(B.8)}
\end{aligned}$$

while (B.6) is rewritten by,

$$y^{(0)}_{rr+2} = \frac{1 - x_{r+1}}{1 - x_{r+1}x_r} \frac{1 - x_{r+1}x_r x_{r+2}}{1 - x_{r+1}x_{r+2}}. \tag{B.9}$$

Eqs.(B.8) and (B.9) lead to $\alpha = x_{r+1}$ and hence we obtain,

$$y^{(0)}_{rr+3} = \frac{1 - x_{r+1}x_{r+2}}{1 - x_r x_{r+1}x_{r+2}} \frac{1 - x_r x_{r+1}x_{r+2}x_{r+3}}{1 - x_{r+1}x_{r+2}x_{r+3}}. \tag{B.10}$$

Similarly we can determine $y^{(0)}_{rr+4}$ by eqs.(B.4) and (B.10) and so on. Then we obtain,

$$y^{(0)}_{rs} = \frac{1 - c_{s-1\,r}}{1 - c_{sr}} \frac{1 - c_{s\,r-1}}{1 - c_{s-1\,r-1}}, \quad (1 \le r < s - 1 < N\,,\, (r, s) \ne (1, N))\,, \tag{B.11}$$

where

$$c_{sr} = x_{r+1}x_{r+2} \cdots x_{s-1}x_s\,. \tag{B.12}$$

In a similar way, we get,

$$\begin{aligned} y^{(0)}_{sr} &= f(x_{r-1}; x_s, y^{(0)}_{sr-1}, 0, x_r) \\ &= \frac{1 - (w/c_{s-1\,r})}{1 - (w/c_{sr})} \frac{1 - (w/c_{s\,r-1})}{1 - (w/c_{s-1\,r-1})}, \quad (1 \le r < s - 1 < N\,,\, (r, s) \ne (1, N)), \end{aligned} \tag{B.13}$$

where

$$w = \prod_{i=1}^{N} x_i\,. \tag{B.14}$$

We should notice that $y^{(0)}_{r+1\,r}$ can be regarded as a diagonal of the quadrilateral $p_{r+1}p_{r-1}p_r o_1$ whose sides are $(y^{(0)}_{r+1\,r-1}, 0, x_r, x_{r+1})$ [11] and hence we obtain,

$$y^{(0)}_{r+1\,r} = f(x_{r-1}; x_{r+1}, y^{(0)}_{r+1\,r-1}, 0, x_r)\,. \tag{B.15}$$

Then eqs.(B.13) and (B.15) lead to

$$\begin{aligned} y^{(0)}_{r+1\,r} &= \frac{1 - w}{1 - (w/c_{r+1r})} \frac{1 - (w/c_{r+1\,r-1})}{1 - (w/c_{r\,r-1})} \\ &= \frac{1 - w}{1 - (w/x_{r+1})} \frac{1 - (w/x_r x_{r+1})}{1 - (w/x_r)}, \quad (0 \le r < N,\ x_0 \equiv x_N). \end{aligned} \tag{B.16}$$

Furthermore $u^{(0)}_{r+1}$ can be regarded as a diagonal of the "quadrilateral" $p_{r+1}p_r p_{r+1}o_1$ whose sides are $(y^{(0)}_{r+1\,r}, 0, x_{r+1}, x_{r+1})$ [11] and hence we obtain,

$$u^{(0)}_{r+1} = f(x_r; x_{r+1}, y^{(0)}_{r+1\,r}, 0, x_{r+1})\,. \tag{B.17}$$

Then eqs.(B.17) and (B.16) lead to

$$u^{(0)}_r = \frac{(1 - x_r w)(1 - (w/x_r))}{(1 - w)^2}, \quad (1 \le r \le N). \tag{B.18}$$

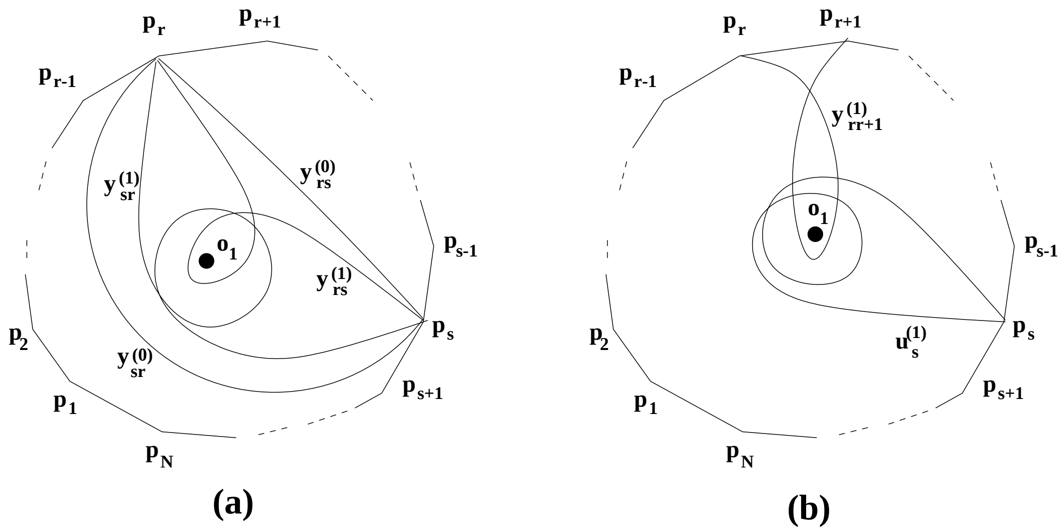

Figure 11: Winding lines in (dual) diagram.

We have seen that there are two topologically inequivalent lines, $y_{rs}^{(0)}$and $y_{rs}^{(0)}$, connecting a pair of vertices of the N-polygon in general. In fact there are infinite number of lines. In Figure 11(a) we find that a line $y_{rs}^{(1)}$ which connects two vertices, p_r and p_s, with winding the point o_1 is topologically inequivalent to $y_{rs}^{(0)}$. Note that $y_{rs}^{(1)}$ is drawn clockwise from p_r to p_s with the point o_1 always kept on the right and hence $y_{rs}^{(1)} \neq y_{sr}^{(1)}$. Then we have infinite number of inequivalent lines $y_{rs}^{(n)}, y_{sr}^{(n)}$ $(n = 0, 1, 2, \cdots)$ connecting a pair of vertices, p_r and p_s $(1 \leq r < s-1 < N)$. We know that $y_{rs}^{(0)}$ is a diagonal of a quadrilateral $o_1 p_r p_{s-1} p_s$ whose sides are $(x_r, y_{rs-1}^{(0)}, 0, x_s)$ and we may regard all the $y_{rs}^{(n)}$ as diagonals of the same quadrilateral $o_1 p_r p_{s-1} p_s$. However, since $y_{rs}^{(n)}$ crosses all the x_i n times, we regard the other diagonal of the quadrilateral as $x_{s-1} w^n$ and hence we get $(n = 0, 1, 2, \cdots)$,

$$\begin{aligned} y_{rs}^{(n)} &= f(x_{s-1} w^n; x_r, y_{rs-1}^{(0)}, 0, x_s) \\ &= \frac{1 - c_{s-1\,r} w^n}{1 - c_{sr} w^n} \frac{1 - c_{s\,r-1} w^n}{1 - c_{s-1\,r-1} w^n}, \qquad (1 \leq r < s-1 < N\,,\ (r,s) \neq (1,N)). \end{aligned} \tag{B.19}$$

Similarly we obtain $(n = 0, 1, 2, \cdots)$,

$$\begin{aligned} y_{sr}^{(n)} &= f(x_{r-1} w^n; x_s, y_{sr-1}^{(0)}, 0, x_r) \\ &= \frac{1 - (w^{n+1}/c_{s-1\,r})}{1 - (w^{n+1}/c_{sr})} \frac{1 - (w^{n+1}/c_{s\,r-1})}{1 - (w^{n+1}/c_{s-1\,r-1})}, \\ &\qquad\qquad (1 \leq r < s-1 < N\,,\ (r,s) \neq (1,N)), \end{aligned} \tag{B.20}$$

$$\begin{aligned} y_{r+1\,r}^{(n)} &= f(x_{r-1} w^n; x_{r+1}, y_{r+1\,r-1}^{(0)}, 0, x_r) \\ &= \frac{1 - w^{n+1}}{1 - (w^{n+1}/x_{r+1})} \frac{1 - (w^{n+1}/x_r x_{r+1})}{1 - (w^{n+1}/x_r)}, \quad (0 \leq r < N,\ x_0 \equiv x_N), \end{aligned} \tag{B.21}$$

$$\begin{aligned} u_r^{(n)} &= f(x_{r-1} w^n; x_r, y_{r\,r-1}^{(0)}, 0, x_r) \\ &= \frac{(1 - x_r w^{n+1})(1 - (w^{n+1}/x_r))}{(1 - w^{n+1})^2}, \quad (1 \leq r \leq N). \end{aligned} \tag{B.22}$$

We have seen that non-winding lines connecting a pair of vertices are accompanied with winding lines and this is also the case for the sides of the N-polygon. In Figure 11(b) we

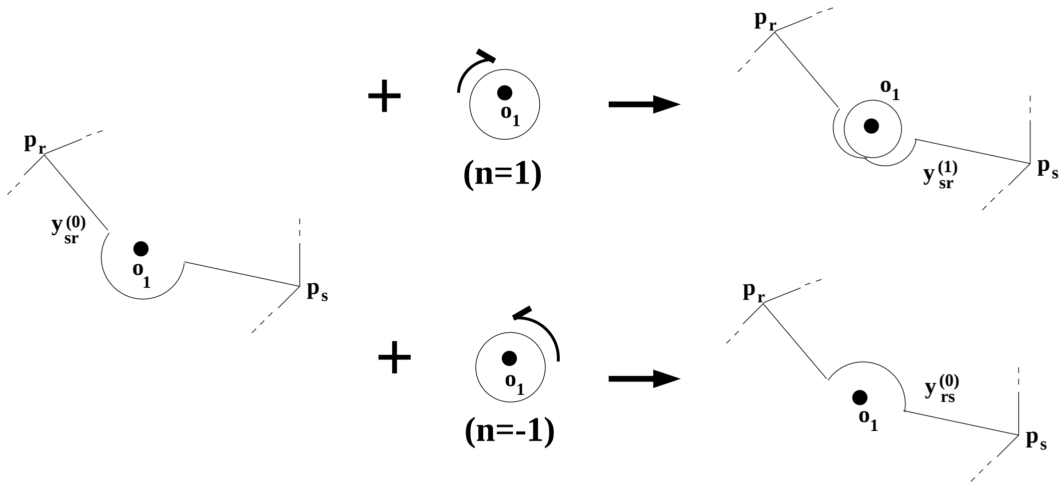

Figure 12: Relations between lines in (dual) diagram.

see that $y^{(1)}_{rr+1}$ is another topologically inequivalent line to those we have seen so far, and the corresponding non-winding line is a side of the N-polygon, $y^{(0)}_{rr+1}(=0)$. We can obtain $y^{(n)}_{rr+1}\,(n=1,2,\cdots)$ as follows. Figure 12 indicates that $y^{(0)}_{sr}$ with a clockwise winding gives $y^{(1)}_{sr}$ while $y^{(0)}_{sr}$ with a counterclockwise winding gives $y^{(0)}_{rs}$, which implies

$$y^{(-1)}_{sr} = y^{(0)}_{rs}\,. \tag{B.23}$$

Assuming that eq.(B.20) holds for $n<0$ we can easily prove eq.(B.23) or we obtain,

$$y^{(-1-n)}_{sr} = y^{(n)}_{rs}\,. \tag{B.24}$$

Then we get $(n=1,2,\cdots)$,

$$\begin{aligned} y^{(n)}_{rr+1} &= y^{(-1-n)}_{r+1r} = \frac{1-w^{-n}}{1-(w^{-n}/x_{r+1})}\,\frac{1-(w^{-n}/x_r x_{r+1})}{1-(w^{-n}/x_r)} \\ &= \frac{1-w^{n}}{1-x_{r+1}w^{n}}\,\frac{1-x_r x_{r+1}w^{n}}{1-x_r w^{n}}\,, \quad (0\le r<N,\, x_0\equiv x_N)\,. \end{aligned} \tag{B.25}$$

Three comments are in order: i) (B.22) satisfies

$$u^{(-2-n)}_r = u^{(n)}_r\,, \quad (n=0,1,2,\cdots). \tag{B.26}$$

ii) Plugging $n=0$ in (B.25) we obtain $y^{(0)}_{rr+1}=0$. iii) If we regard $y^{(0)}_{rr+1}$ as a diagonal of a "quadrilateral" $p_r p_r p_{r+1} o_1$ whose sides are $(x_r, u^{(-1)}_r, 0, x_{r+1})$ and the other diagonal is x_r (Figure 13), we have,

$$y^{(0)}_{rr+1} = f(x_r; x_r, u^{(-1)}_r, 0, x_{r+1})\,, \tag{B.27}$$

or

$$y^{(0)}_{rr+1} = \frac{1-x_r\alpha}{1-x_r\alpha x_r}\,\frac{1-x_r\alpha x_r x_{r+1}}{1-x_r\alpha x_{r+1}}\,, \tag{B.28}$$

$$u^{(-1)}_r = \frac{(1-\alpha)(1-\alpha x_r^2)}{(1-\alpha x_r)^2}\,. \tag{B.29}$$

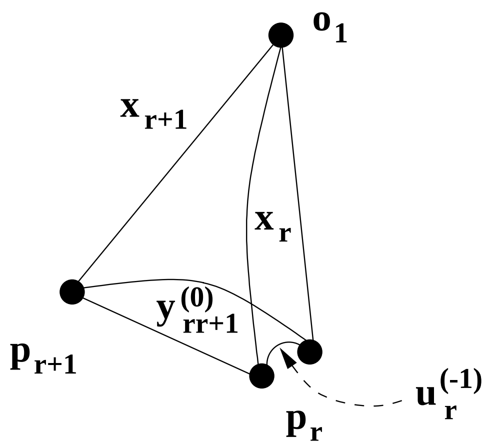

Figure 13: A "quadrilateral". $x_r, y^{(0)}_{rr+1}$ and p_r are written double to be explained.

From eq.(B.22) with $n = -1$ and eq.(B.29) we have $\alpha = 1/x_r$, and hence (B.28) becomes $y^{(0)}_{rr+1} = 0$. Then eq.(B.27) leads to ($n = 1, 2, \cdots$)

$$\begin{aligned} y^{(n)}_{rr+1} &= f(x_r w^n; x_r, u^{(-1)}_r, 0, x_{r+1}) \\ &= \frac{1 - x_r w^n(1/x_r)}{1 - x_r w^n(1/x_r)x_r} \frac{1 - x_r w^n(1/x_r)x_r x_{r+1}}{1 - x_r w^n(1/x_r)x_{r+1}} \\ &= \frac{1 - w^n}{1 - x_r w^n} \frac{1 - x_r x_{r+1} w^n}{1 - x_{r+1} w^n}, \end{aligned} \tag{B.30}$$

which agrees with (B.25).

Since we have obtained all the topologically inequivalent lines, we can now write down the N-point one-loop amplitude by the KSV method. The square of the momentum associated with a line depends only on the ends of the line and hence, for example, the exponent for $y^{(n)}_{rs}$ is the same as that for $y^{(n)}_{sr}$ and it is given by $-\alpha(-(p_r - p_s)^2) - 1$, where α is given by eq.(3.7). The N-point one-loop amplitude is given by,

$$\begin{aligned} A^{(1)}_N &= g^N \int d^{26}k \int_0^1 \prod_{i=1}^N dx_i \, \rho^{(1)}(x_i) \prod_{i=1}^N x_i^{-\alpha(-(k-p_i)^2)-1} \prod_{\substack{1\le r<s-1<N \\ (r,s)\neq(1,N)}} \left(\prod_{n=0}^\infty y^{(n)}_{rs}\, y^{(n)}_{sr} \right)^{-\alpha(-(p_r-p_s)^2)-1} \\ &\times \prod_{r=0}^{N-1} \left(\prod_{n=0}^\infty y^{(n)}_{r+1r} \prod_{n=1}^\infty y^{(n)}_{rr+1} \right)^{-\alpha(-(p_r-p_{r+1})^2)-1} \prod_{r=1}^N \left(\prod_{n=0}^\infty u^{(n)}_r \right)^{-\alpha(0)-1}, \end{aligned} \tag{B.31}$$

where $\rho^{(1)}(x_i)$ is given by,

$$\rho^{(1)}(x_i) \propto \prod_{1\le r<s\le N} (1 - x_{r-1}x_r)^{-1}, \quad (x_0 \equiv x_N), \tag{B.32}$$

due to crossing symmetry. The amplitude in the operator formalism is given by (see e.g.[13]),

$$\begin{aligned} A^{(1)}_{Nop} &= g^N \int d^{26}p \int_0^1 \prod_{i=1}^N dX_i \left(\frac{1}{W^2}\right) \prod_{i=1}^N X_i^{\alpha' q_i^2} \prod_{n=1}^\infty (1 - W^n)^{-24} \\ &\times \prod_{1\le r<s\le N} \left\{ \prod_{n=1}^\infty \exp\left(-\frac{C^n_{sr} + (W/C_{sr})^n - 2W^n}{n(1 - W^n)} \right) \right\}^{P_r\cdot P_s} \end{aligned}$$

$$= g^N \int d^{26}p \int_0^1 \prod_{i=1}^N dX_i \prod_{i=1}^N X_i^{-\alpha(-(p+p_N-p_{i-1})^2)-1} \prod_{n=1}^\infty (1-W^n)^{-24}$$
$$\times \prod_{1\le r<s\le N} \left\{ (1-C_{sr}) \prod_{n=1}^\infty \left(\frac{(1-C_{sr}W^n)(1-W^n/C_{sr})}{(1-W^n)^2} \right) \right\}^{P_r\cdot P_s}, \qquad \text{(B.33)}$$

where

$$q_r = p - P_1 - P_2 - \cdots - P_{r-1} = p + p_N - p_{r-1}, \qquad \text{(B.34)}$$
$$C_{sr} = X_{r+1}X_{r+2}\cdots X_{s-1}X_s, \qquad \text{(B.35)}$$
$$W = X_1X_2\cdots X_N. \qquad \text{(B.36)}$$

Considering the correspondence of the parameters,

$$\begin{aligned} k &\longleftrightarrow p+p_N, \\ x_i &\longleftrightarrow X_{i+1}, \\ c_{sr} &\longleftrightarrow C_{s+1r+1}, \\ w &\longleftrightarrow W, \end{aligned}$$

we shall compare the amplitudes. Then we find,

$$\rho^{(1)}(x_i) = \left[\prod_{n=1}^\infty (1-w^n) \right]^{-24} \prod_{1\le r<s\le N} (1-x_{r-1}x_r)^{-1}. \qquad \text{(B.37)}$$

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